AFRL-WS-WP-TM-1999-9005

THE INVESTIGATION OF STRUCTURAL MEMBERS UNDER COMBINED AXIAL AND TRANSVERSE LOADS
Section I

Air Service Information Circular, Volume V, No. 493

J. S. NEWELL

Air Service
Engineering Division
McCook Field
Dayton OH 45430

December 1, 1924

AIR FORCE RESEARCH LABORATORY
AIR FORCE MATERIEL COMMAND
WRIGHT-PATTERSON AIR FORCE BASE OH 45433

REPORT DOCUMENTATION PAGE			*Form Approved OMB No. 0704-0188*

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1, 1924

3. REPORT TYPE AND DATES COVERED
FINAL August 7, 1924

4. TITLE AND SUBTITLE
THE INVESTIGATION OF STRUCTURAL MEMBERS UNDER COMBINED AXIAL AND TRANSVERSE LOADS Section I
Air Service Information Circular, Volume V, No. 493

5. FUNDING NUMBERS

6. AUTHOR(S)
J. S. NEWELL

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Service
Engineering Division
McCook Field
Dayton OH 45430

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Service
Engineering Division
McCook Field
Dayton OH 45430

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
AFRL-WS-WP-TM-1999-9005

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION AVAILABILITY STATEMENT
Approved for public release; Distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
Historical study that provides a "satisfactory "method of computing the stresses in such members of an airplace structure as are subjected to combined axial and transverse. The purpose of the investigation was to make a comparative study of a number of approximate methods for the design of members under combined loads to ascertain their relative degrees of conservatism and ease of application.

14. SUBJECT TERMS
Stress loads, Axial Loads, Transverse Loads

15. NUMBER OF PAGES
28

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

File D 52.16 / 63

McCOOK FIELD REPORT, SERIAL No. 2400

AIR SERVICE INFORMATION CIRCULAR

(AVIATION)

PUBLISHED BY THE CHIEF OF AIR SERVICE, WASHINGTON, D. C.

Vol. V | December 1, 1924 | No. 493

THE INVESTIGATION OF STRUCTURAL MEMBERS UNDER COMBINED AXIAL AND TRANSVERSE LOADS

SECTION I

(AIRPLANE SECTION REPORT)

▽

Prepared by J. S. Newell
Engineering Division, Air Service
McCook Field, Dayton, Ohio
August 7, 1924

WASHINGTON
GOVERNMENT PRINTING OFFICE
1925

This report covers Parts I and II of the data resulting from an investigation of structural members subjected to combined axial and transverse loads. It contains the theoretical portion of the development of the United States Army Air Service formulas and includes tables of functions which greatly facilitate the use of these formulas.

Studies are now being made as to the agreement of the deflections computed by the use of these formulas with the deflections actually obtained by tests made on experimental struts and trusses, it being considered that a satisfactory check on the dependability of the formulas can be obtained in this way. When these studies, which have now been carried far enough to indicate that the formulas are reliable, are completed, Section II of this report will be issued.

The reason for publishing this report in two sections is to furnish the airplane industry immediately with a more complete set of data on the development and use of the precise formulas than is now available. This section is self-contained in that it includes all information necessary to the intelligent use of the precise formulas in the design of airplane structures.

THE INVESTIGATION OF STRUCTURAL MEMBERS UNDER COMBINED AXIAL AND TRANSVERSE LOADS

PURPOSE OF THE INVESTIGATION

The purpose of the investigation, the results of which are given in this report, was to obtain a satisfactory method of computing the stresses in such members of an airplane structure as are subjected to combined axial and transverse loads.

SCOPE OF THE INVESTIGATION

The investigation was initiated for the purpose of making a comparative study of a number of approximate methods for the design of members under combined loads to ascertain their relative degrees of conservatism and ease of application. Several approximate methods were studied and applied to a specific problem for purposes of comparison. A full discussion of the methods and results obtained will be found in Part I of this report.

Although several of the approximate formulas were found to agree quite well among themselves, there were some that gave results which were considerably different, so it was decided to investigate the precise formulas developed by Mr. Arthur Berry in his paper, "The Calculation of Stresses in Aeroplane Wing Spars,"[1] and to compare the results thus obtained with those from the approximate methods. It was found that considerable discrepancy existed between the results, the approximate methods being too conservative in some cases and unsafe in others.

Attention was then directed toward the precise methods of Berry and of Muller-Breslau [2] and an effort was made to simplify them and, if possible, to reduce the purely mechanical part of the mathematical work inherent in their use. Part II is devoted to the development of different forms of the precise formulas for the different conditions of loading that are encountered in airplane structural design. It is entirely theoretical in its treatment of members under combined loading.

Parts III and IV are devoted to tests made on spruce specimens subjected to combined loads for the purpose of checking the theory and the formulas developed in Part II of this report.

Part III is concerned with simple, pin-ended struts. Part IV, with continuous members.

The appendix contains tables for use with the formulas developed in Part II and articles on subjects closely related to the formulas, their development and use.

[1] Trans. Royal Aeronautical Society, London, 1919.

[2] Graphische Statik, Vol. II, part 2.

SUMMARY OF THE RESULTS OF THIS INVESTIGATION

The study of the various approximate methods, the results of which are given in Part I, indicated that the combination of the ordinary three-moment equation, which does not provide for an axial load, with the various approximate formulas gave results that would not be dependable on a structure such as an airplane spar. As a result of this, the precise formulas of Berry and Muller-Breslau were investigated for the purpose of reducing the labor involved in their application. The formulas developed in Part II will be found to be somewhat less laborious than those of Berry and, with the use of the special tables of sines, cosines, and tangents, somewhat easier than Muller-Breslau's method. Formulas for loading conditions other than those given by Berry or Muller–Breslau are also developed in Part II, so that precise formulas are now available for all loading conditions liable to occur in airplane design.

The tests which are described in Part III were sufficient to establish the theory upon which the precise methods depend, but since they were made on small specimens under laboratory conditions they did not indicate how dependable the precise methods would be under practical conditions. The tests which are discussed in Part IV were made on a small truss similar to the lift truss of an airplane and indicate that for continuous members under combined load the precise formulas are more dependable than the approximate. The precise formulas gave results within 5 per cent of those indicated by the tests where the ordinary methods were as much as 16 per cent off.

The conclusions obtained from this investigation are as follows:

The precise formulas of Muller-Breslau and Berry accurately represent the forces and stresses in members subjected to combined axial and lateral loads.

The theoretical loading conditions for which these formulas are developed are sufficiently close to the actual conditions that the formulas may be used safely in practical design.

The precise formulas developed in this investigation are fundamentally identical with the Muller-Breslau and Berry formulas, but are easier to apply in practical design and cover more conditions of loading.

In fact, in many cases these precise formulas are easier to apply than the best approximate formulas.

The approximate formulas studied, and they include all those in common use, are too inaccurate for general employment and may give very unsafe results.

The approximate formulas may be used in preliminary design and in final design also if the secondary stresses and slenderness ratio are small and the margin of safety is large.

The precise formulas must be used in design if the secondary stresses or slenderness ratio is large or the margin of safety is small

THE ACTION OF A MEMBER UNDER COMBINED AXIAL AND TRANSVERSE LOAD

A concise statement of the action of a strut under combined loads is desirable to clarify the problem entailed, as well as the methods for its solution which have been investigated or developed in this report.

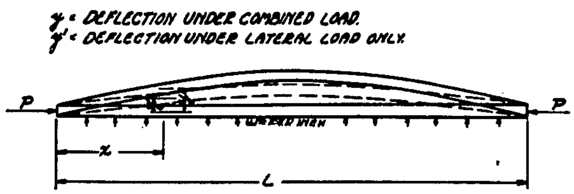

FIG. 1.

Figure 1 shows a beam acted upon by a side load of w pounds per inch and an axial load of P pounds. Due to the side load alone the beam deflects a distance y' at a point x inches from the left support and the bending moment at that point is $\frac{wx^2}{2} - \frac{wLx}{2}$.

If a compressive load, P, is applied as shown, the moment at point x will be increased by $-Py'$, since the load P acts at a distance y' from the axis of the deflected member. This increase in moment causes a greater deflection at x, which, in turn, causes a further increase in the moment. If the load P is not great enough to cause failure, these increments of the moment and deflection will get smaller and smaller until the strut comes into equilibrium and the deflection at x becomes y. If, however, the load P is sufficiently large, the increments of deflection will be successively greater and greater until the strut fails by buckling.

On the other hand, if P be tension instead of compression the deflection y' due to the side load alone will be reduced instead of increased and, as P is increased, the strut will tend to straighten out and the moment at any point will be reduced. The failure, when it occurs, will be a tension failure and will not be accompanied by buckling of the member.

It is apparent from the above discussion that an axial compressive load, which increases the bending moment at every point, is of far greater importance in the design of members under combined loads than is an axial tension which tends to decrease the bending. For this reason the investigation has been confined almost entirely to the case of a compressive load, although some attention has been given to axial tension and methods of providing for it.

It is apparent that the increments to the deflection or bending moment should be represented by a series of some sort which, if the axial load is not too great, will converge, so that the limit of the series may be taken to represent the condition when the member comes to rest and is in equilibrium. If the member is continuous over two or more supports, the moments will be increased, both at the supports and in the spans, by the application of the axial load. The ordinary three-moment equation, which would be used on a continuous beam in conjunction with the various approximate methods for computing stresses under combined loading, makes no provision for the change in moments over the supports due to the axial load, and so vitiates the effect of any series or other device used in the formulas to provide for the effects of the axial load in the spans. This is an important point, as it accounts for a large part of the discrepancy between the approximate and the precise methods when applied to continuous members, such as the wing spars of an airplane.

The precise methods provide for the axial load both in the three-moment equation and in the formulas for the moment in the spans by the use of mathematical series. It so happens that the series used are identical with those of the trigonometric functions, sines, cosines, and tangents, but it should be borne in mind that they are not connected with angles in any way. The same results could be obtained by substituting the series for the sines or cosines in the formulas, but since the limits of these series have already been computed and tabulated so that it is far simpler to determine the value of the sine from a table than to determine the limit of the series it represents, the terms "sine" and "cosine" are used in the precise formulas. Special tables have been computed for sines, cosines, and tangents for a range of the variable from 0 to 3.50. If desired, the variable may be considered to be an angle expressed in radians, and the sine, cosine, or tangent may be obtained from any set of trigonometric tables for this angle converted to degrees and minutes.

PART I—COMPARISON OF APPROXIMATE METHODS

The six approximate methods investigated and recorded in this report were studied by applying the various formulas to one specific spar and loading. The results are tabulated and compared and will be found in detail in the following pages.

Figure 2 shows the member on which the computations were made. It is a section of a wing beam from the Boeing GA–2 airplane continuous over two spans. It is subjected to a uniformly distributed transverse load, axial loads, and a restraining moment at one end caused by the continuity of the beam and a cantilever overhang. The other end is pinned. The moment at the intermediate support was found by the ordinary three-moment equation which does not provide for the effects of an axial load.

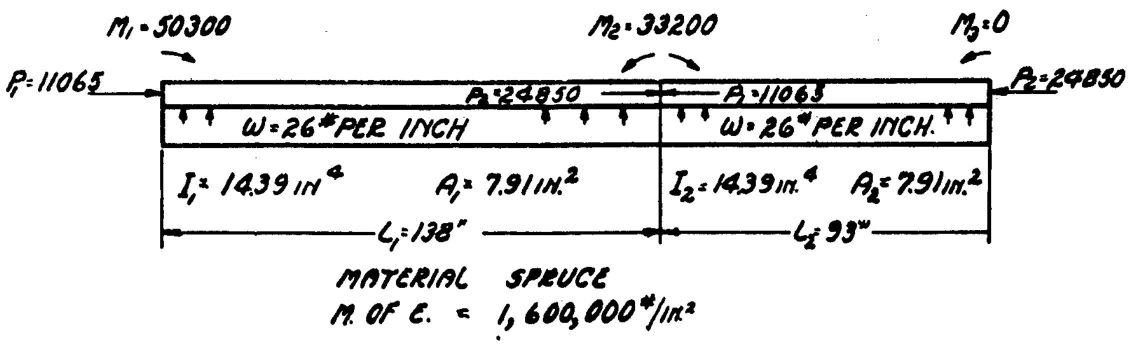

FIG. 2

It will be noted that the lateral load shown in Figure 2 acts upward in accordance with the conventions used in airplane design. All of the approximate formulas have been derived for bridge or building structures in which the loads act downward. This difference in direction of loading causes considerable difficulty in the matter of signs and great care is required when employing the approximate formulas to use the proper signs.

The formula given in this report will be modified where necessary to conform to the conventions used in airplane structural design, which are as follows:

Forces are considered positive when acting upwards; shear, when the algebraic sum of all forces acting on the beam to the left of the section considered is positive; bending moments, when they tend to cause compression in the upper fibers of the part of the beam to the right of the section; the slope of a line, when it rises from left to right; and deflection, when the deflected position of a point is above the original position.

THE FIRST APPROXIMATE METHOD

One of the best known of the approximate formulas for computing the bending moment in a member subjected to combined axial and lateral load is that given on page 520, Volume II, of Johnson, Bryan, and Turneaure's "Modern Framed Structures." For a beam having hinged ends it is,

$$M_{\max.}=\frac{M_o}{1\pm\frac{P(L')^2}{10\,EI}}$$

where M_o is the maximum moment with no axial load and L' is the distance between hinge points. The negative sign in the denominator is used when the axial load causes compression, the positive when it causes tension.

This formula is developed for pin-ended members, but can be applied to any member by considering the sections beween points of inflection as simple, pin-ended spans. Where M_1 and M_2, the restraining moments at the points of support, have been determined, the distance between the points of inflection may be found, the axial load being neglected, from

$$L'=2\sqrt{\left(\frac{M_2-M_1}{wL}\right)^2-\frac{M_2+M_1}{w}+\frac{L^2}{4}}$$

in which w is the lateral load per inch run and L is the distance between supports. This expression is applicable to a uniformly distributed load and similar expressions can be developed for other loading conditions. The value of L' determined from this equation is in error, due to the fact that the points of inflection move somewhat under the action of the axial load, which is neglected entirely in this expression. The effect of this error is generally not great, so that the formula may be used for preliminary analyses, etc.; but its presence should be noted and borne in mind.

M_o may be found from $\frac{-w(L')^2}{8}$, i. e., the moment at mid-span on a simple beam under a uniformly distributed load.

The application of this method to the beam of Figure 2 gives the following results:

For the 138-inch span the distance between points of inflection is

$$L'=2\sqrt{\left(\frac{M_2-M_1}{wL}\right)^2-\frac{M_2+M_1}{w}+\frac{L^2}{4}}=2$$

$$\sqrt{\left(\frac{33{,}200-50{,}300}{26\times 138}\right)^2-\frac{33{,}200+50{,}300}{26}+\frac{138^2}{4}}=79.2''$$

The maximum moment between points of inflection will be

$$M_o=\frac{-w(L')^2}{8}=\frac{-26\times 79.2^2}{8}=-20{,}450 \text{ in. lbs.}$$

$$M_{1-2}=\frac{M_o}{1-\frac{P(L')^2}{10\ EI}}$$

$$=\frac{-20{,}450}{1-\frac{11{,}065\times(79.2)^2}{10\times 1{,}600{,}000\times 14.39}}=-29{,}300 \text{ in. lbs.}$$

Similarly, for the 93-inch bay

$$L'=2\sqrt{\left(\frac{0-33{,}200}{26\times 93}\right)^2+\frac{33{,}200}{26}+\frac{93^2}{4}}=65.4 \text{ in. be-}$$

tween points of inflection.

$M_{2-3}=-26{,}200$ in. lbs.

This formula is sometimes given as $M_{max.}=\dfrac{M_o}{1-\frac{P(L')^2}{\pi^2\ EI}}$ or $M_{max.}=\dfrac{M_oQ}{Q-P}$ where Q is the Euler load for the part of the spar between the points of inflection. It will be noted that the difference between this formula and Johnson's is that $\frac{1}{\pi^2}$ has been substituted for $\frac{1}{10}$ in the second term of the denominator. A similar formula may be derived from the precise equation for the maximum moment in a pin-ended member under combined load, if the series represented by the sines and cosines are substituted for the functions themselves. When the resulting expression is simplified, neglecting terms containing powers of E and I, it becomes

$$M_{max.}=\frac{M_o}{1-\frac{5P\ (L')^2}{48\ EI}}$$

which reduces to Johnson's formula on substituting $\frac{1}{10}$ for $\frac{5}{48}$. These last two formulas are applicable only to the case where the axial load is compression. Of the three formulas, the last is the most conservative, although the difference between them is slight.

The maximum moments in the spans of a continuous beam subjected to combined load are somewhat in error when computed by this method. The greatest source of error is the fact that the ordinary three-moment equation does not provide for the effect of the axial load, so that the computed moments at the supports are less than they should be. This results in the computed moments in the spans being incorrect, since it affects the location of the points of inflection. A second source of error is the fact that the points of inflection move under the influence of the axial load, an effect which is also neglected in these formulas.

This method is sufficiently accurate for use in preliminary analyses or designs of airplanes, but it should not be depended upon for the final design of continuous or restrained members under combined load. Johnson's formula, or either of the others, will give satisfactory results for pin-ended struts having a side load and it may safely be used for the design of such members.

THE SECOND APPROXIMATE METHOD

A second well-known formula for the maximum moment in a pin-ended member under a combined lateral and axial load is the so-called secant formula, $M_{max.}=M_o\pm Py\left(\sec\frac{L}{2}\sqrt{\frac{P}{EI}}\right)$. The positive sign is used when the axial load is compressive. This formula may be applied to continuous members in the same way that Johnson's formula was, i. e., by computing the distance L' between the points of inflection and substituting for L. In this case $M_o=\frac{-w(L')^2}{8}$ for a uniformly distributed lateral load and $y=\frac{-5w(L')^4}{384\ EI}$. For other conditions of loading the values of M_o and y may be found similarly from the corresponding expressions for the moment and deflection at the mid-point of a beam of length L' under the given loading.

In this formula the primary deflection, y, is multiplied by an infinite series, the limit of which is given by the secant of the quantity, $\frac{L}{2}\sqrt{\frac{P}{EI}}$. This is done to obtain the deflection when the strut comes into equilibrium under the combined load. This ultimate deflection is then multiplied by the axial load P to obtain the secondary moment and the latter quantity is added to the primary bending moment to obtain the total.

This formula is susceptible to the same criticism as Johnson's when applied to continuous or restrained members. It gives satisfactory results when applied to pin-ended struts with lateral loads, but somewhat more labor is involved in its use than is required with Johnson's formula.

When applied to the spar and loading shown in Figure 2, M_o, P, L', E, and I are the same as before.

For the 138-inch span

$$y=\frac{-5w\ (L')^4}{384\ EI}=\frac{-5\times 26\times(79.2)^4}{384\times 1{,}600{,}000\times 14.39}=-0.579 \text{ in.}$$

$$\frac{L'}{2}\sqrt{\frac{P}{EI}}=\frac{79.2}{2}\sqrt{\frac{11{,}065}{1{,}600{,}000\times 14.39}}=0.87 \quad \text{Sec } 0.87$$

$$=\frac{1}{\cos 0.87}=\frac{1}{0.6448}. \text{ (See tables in Appendix.)}$$

$$M_{1-2}=-20{,}450+\frac{11{,}065\ (-0.579)}{0.6448}=-20{,}450$$
$$-9{,}900=30{,}350 \text{ in. lbs.}$$

Similarly, for the 93-inch span

$$M_{2-3}=-27{,}850 \text{ in. lbs.}$$

THE THIRD APPROXIMATE METHOD

A somewhat similar formula was used by the Forest Products Laboratory in the development of a wing beam for the Navy type TB flying boat. The formula is $M_{max.}=M_o+\dfrac{1.2\,Py}{1-\dfrac{P}{Q}}$ where y is the primary deflection due to the lateral load at the point of maximum moment and Q is the allowable Euler load on the section between pin points or between points of inflection on a continuous or restrained beam. This formula is applicable to members having an axial compressive load and it is open to the same errors as are formulas 1 and 2, which depend on the distance between the points of inflection computed without providing for the effect of the axial load.

M_o, P, y, L', E, and I are the same as the values used with formula 2, so that, when this formula is applied to the beam shown in Figure 2, the results are:

For the 138-inch span,

$$Q=\frac{\pi^2EI}{(L')^2}=\frac{\pi^2\times1{,}600{,}000\times14.39}{(79.2)^2}=36{,}200\text{ lbs.}$$

$$M_{1-2}=M_o+\frac{1.2\,Py}{1-\dfrac{P}{Q}}=-20{,}450+\frac{1.2\times11{,}065(-0.579)}{1-\dfrac{11{,}065}{36{,}200}}$$

$$=-20{,}450-11{,}050=-31{,}500\text{ in. lbs.}$$

Similarly, for the 93-inch bay,

$$M_{2-3}=-28{,}900\text{ in. lbs.}$$

THE FOURTH APPROXIMATE METHOD

On page 155 of "Flugzeugstatik," Van Gries gives an approximate formula for determining the maximum moment in the span of a beam under combined loading. This formula is $M_{max.}=\dfrac{-0.16\,wL^2}{1-\dfrac{0.16\,PL^2}{EI}}$, where L is one-half the span length. Van Gries derives this from the so-called exact cosine formula for fixed ended beams. It is admittedly approximate and is probably none too reliable. When applied to the beam shown in Figure 2 this formula gives the following results:

For the 138-inch bay

$$M_{max.}=\frac{-0.16\times26\times69^2}{1-\dfrac{0.16\times11{,}065\times69^2}{1{,}600{,}000\times14.39}}=\frac{-19{,}800}{0.634}$$

$$=-31{,}200\text{ in. lbs.}$$

For the 93-inch bay

$$M_{max.}=-14{,}300\text{ in. lbs.}$$

Van Gries questions the dependability of this formula, but it appears to give fairly good results in this case. The maximum moment in the long bay agrees quite closely with that obtained by the other approximate formulas, while that in the shorter span is in fairly good agreement with the results of the precise methods. No provision is made for different degrees of restraint at the supports of a continuous beam, since none of the terms in this formula depend on these moments or upon the points of inflection computed from them.

If the distances between points of inflection had been used instead of the total lengths of the bays, the maximum moments in the spans would have been only $-7{,}430$ in. lbs. in the 138-inch and $-5{,}460$ in. lbs. in the 93-inch bay. These results do not agree with those obtained by any of the other methods and are probably much too small.

It is not recommended that this formula be used for the design of airplane members subjected to combined loads, as it is believed that the good results on the first example above are purely accidental.

THE FIFTH APPROXIMATE METHOD

An approximate method that has been much used in airplane design is that shown in Chapter III of "Structural Analysis and Design of Airplanes." Briefly, this method consists in assuming that the maximum total moment occurs at the point of zero shear, computing the deflection of the point of zero shear due to the lateral load only, multiplying this deflection by the axial load, and adding the result to the moment at that point due to the lateral load alone. This method is in error for three reasons: It depends on values for the moments at the supports which are computed without considering the effect of the axial load; the point of maximum total moment and the point of maximum moment due to side load only are assumed to be identical; and only the first of the infinite series of secondary moments is considered.

Applying this method to the 138-inch bay, the results are as follows:

Shear at left end, $S_{+1}=\dfrac{M_2-M_1}{L}-\dfrac{wL}{2}$

$$=\frac{33{,}200-50{,}300}{138}-\frac{26\times138}{2}=-1{,}918\text{ lbs.}$$

Location of point of zero shear from left end,

$$X=1{,}918/26=73.8\text{ in.}$$

Max. moment due to side load only

$$M_x=M_1+(S_{+1})x+\frac{wx^2}{2}=50{,}300-1{,}918\times73.8$$
$$+13\times73.8^2=-20{,}450\text{ in. lbs.}$$

Deflection at point of zero shear

$$y=\frac{x}{EI}(x-L)\left[\frac{M_1}{2}+\frac{S_{+1}}{6}(x+L)+\frac{w}{24}(x^2+xL+L^2)\right]$$
$$=\frac{73.8(73.8-138.0)}{1{,}600{,}000\times14.39}\left[\frac{50{,}300}{2}-\frac{1{,}918}{6}(73.8+138.0)\right.$$
$$\left.+\frac{26}{24}(73.8^2+73.8\times138+138^2\right]$$
$$=+1.025\text{ in.}$$

Secondary moment $Py=-11{,}065\times1.025=-11{,}350$ in. lbs.

Total moment $M_{max.}=Mx+Py=-20{,}450-11{,}350$ $=-31{,}800$ in. lbs.

Applying the same method to the 93-inch bay, the maximum bending moment is $-22{,}520$ in. lbs.

The value of the maximum deflection might be used in computing the secondary moment instead of the deflection at the point of zero shear, but the difference in the result will seldom be great, and the computations required are very tedious.

THE SIXTH APPROXIMATE METHOD

The United States Army Air Service has sometimes used a method called that of "secondary deflections" for computing the moments in airplane members subjected to combined axial and transverse loads. The method is outlined on pages 67 and 68 of the 1920 edition of "Structural Analysis and the Design of Airplanes." In this method, the axial load being neglected, the bending moment, M_o, and deflection, y_o, are computed at the point of zero shear in the span under consideration. The deflection thus obtained when multiplied by the axial load gives a secondary moment, Py_o. The magnitude of the uniformly distributed load over the entire span, which would give a moment equal to Py_o is then computed, and also the deflection due to such a load. The ratio of this secondary deflection to the primary is used as the constant, r, in a geometric series where the ultimate deflection, when the member comes into equilibrium, is $y=y_o(1+r+r^2+r^3+\ldots\ r^{n-1})$. The total moment is then $M=M_o+Py$ or $M=M_o+M'(1+r+r^2+r^3+\ldots)$, which becomes $M=M_o+\frac{M'}{1.00-r}$.

This method was applied to the beam of Figure 2 with the following results. As far as possible the values obtained by the fifth method were used.

In the 138-inch bay the primary moment at the point of zero shear is $-20{,}450$ in. lbs.

The primary deflection of this point, $y_o=1.025$ in.

The first secondary moment is $Py_o=-11{,}350$ in. lbs.

w', the uniformly distributed load that would give this moment at the center of a simple supported beam 138 inch long is

$$w'=-\frac{8M}{L^2}=\frac{8\times 11{,}350}{138\times 138}=4.77 \text{ lbs. per in.}$$

$$r=\frac{4.77}{26}=0.1834.$$

The maximum moment in the 138-inch span then becomes

$$M=M_o+\frac{M'}{1.00-r}=-20{,}450+\frac{-11{,}350}{1.00-0.1834}$$
$$=-20{,}450-13{,}900=-34{,}350 \text{ in. lbs.}$$

For the 93-inch bay this method gives a value of $M=-26{,}270$ in. lbs.

This method is probably as reliable as any of the approximate formulas, but it, too, is dependent upon the moments at the points of support computed from the ordinary three-moment equation, which does not provide for the axial load. For this reason the assumed point of maximum moment, the point of zero shear, is not correctly located, so that the maximum moment as obtained by this method is in error both as to magnitude and location.

THE PRECISE METHOD

For purposes of comparison the moments at the strut points and in the spans were computed by the formulas developed in Part II of this report.

The first step is to determine the moments at the points of support when the effect of the axial load is provided for. The form of the three-moment equation for this case is

$$\alpha_1 M_1 L_1+2M_2(\beta_1 L_1+\beta_2 L_2)+\alpha_2 M_3 L_2$$
$$=\frac{w_1 L_1^3}{4}\gamma_1+\frac{w_2 L_2^3}{4}\gamma_2$$

Substituting the proper values for α, β, γ, L, etc.,

$$M_2=46{,}100 \text{ in. lbs.}$$

The values for the maximum moments in the bays are $-43{,}400$ in. lbs. for the 138-inch span and $-8{,}900$ in. lbs. for the 93-inch span.

The method of obtaining these values is given in Part II of this report.

WEBB AND THORNE'S METHOD

A method which is similar to this precise method in its derivation but is different from it in that approximate algebraic coefficients are used in place of the exact trigonometric functions occurring in the precise formulas is to be found on page 121 et seq. of Pippard and Pritchard's "Aeroplane Structures." This method is accredited to Messrs. Webb and Thorne, and it gives results that are in close agreement with those obtained by the precise formulas so long as the ratio of P/Q is less than about 0.85. This method permits the computation of the strut point moments by a three-moment equation which provides for the effect of axial loads. The formulas for the maximum moments in the bays also provide for such loads. Webb and Thorne's method has an advantage in that it requires no tables, but the computations involved in its use are somewhat greater than those necessary with the precise formulas given in Part II when the tables are at hand. It is stated that this method will give the strut point moments within one-half of 1 per cent when the ratio of P/Q is less than 0.83, while the probable error in the maximum moment in a span is less than 5 per cent under the same conditions.

The Webb and Thorne equation of three moments for the case of a uniformly distributed side load and an axial compression is

$$\frac{1}{(Q_1-P_1)L_1}\left\{M_1\left(1+0.2\frac{P_1}{Q_1}\right)+2M_2\left(1-0.38\frac{P_1}{Q_1}\right)-\frac{1}{4}w_1 L_1^2\left(1-0.014\frac{P_1}{Q_1}\right)\right\}$$
$$+\frac{1}{(Q_2-P_2)L_2}\left\{M_3\left(1+0.2\frac{P_2}{Q_2}\right)+2M_2\left(1-0.38\frac{P_2}{Q_2}\right)-\frac{1}{4}w_2 L_2^2\left(1-0.014\frac{P_2}{Q_2}\right)\right\}=0$$

where P is the axial load on the spar;

$Q=\frac{\pi^2 EI}{L^2}$;

w=the uniform load, acting upward;

L=the span length.

The expression for the maximum moment in a span is found from

$$M_{\max.}=\frac{Q}{Q-P}\left\{\left(\frac{M_1+M_2}{2}\right)\left(1+0.26\frac{P}{Q}\right)-\frac{1.02\ wL^2}{8}\right\}$$
$$\mp\frac{(M_2-M_1)^2}{2wL^2}$$

The distance from the left-hand support to the point of maximum moment is

$$x=\frac{L}{2}-\frac{M_2-M_1}{wL}$$

Application of this method to the structure used for purposes of comparison gives the following results. The moment at the intermediate strut point, M_2, is found to be 46,100 in. lbs., while the maximum moment in the 138-inch bay is −45,620 in. lbs. For the 93-inch bay the maximum moment is given as −3,570 in. lbs., which is not in very good agreement with the results from the precise equations, while the other two values given above are a very close accord. The ratios of P/Q are high in both of these bays and the results depend on relatively small differences of large numbers, which probably accounts for the differences found. This is particularly true in the case of the shorter span where the difference between two quantities is so small that a slight error in the computations will change the sign of the result from positive to negative.

COMPARISON OF THE VARIOUS RESULTS

Table I gives the values of the moments at the points of support and in the spans of the beam shown in Figure 2 as they were computed by the various methods discussed in the foregoing pages. The value of the moment at the outer support is constant in each method, as it depends on the load on the cantilever tip. The inner end of the beam is pinned to the fuselage, so that M_3 is assumed to be zero in each case. The moment at the intermediate support, M_2, is obtained from the three-moment equation and the moments in the spans by one of the formulas discussed above.

TABLE I

Method	M_1	M_{1-2}	M_2	M_{2-3}	M_3
1	+50, 300	−29, 300	+33, 200	−26, 200	0
2	+50, 300	−30, 350	+33, 200	−27, 850	0
3	+50, 300	−31, 500	+33, 200	−28, 900	0
4	+50, 300	−31, 200	+33, 200	−14, 300	0
5	+50, 300	−31, 800	+33, 200	−22, 520	0
6	+50, 300	−34, 350	+33, 200	−26, 270	0
Precise	+50, 300	−43, 400	+46, 100	−8, 900	0
Webb & Thorne	+50, 300	−45, 620	+46, 100	−3, 570	0

A comparison of the values obtained from the various approximate formulas shows that they agree quite well amongst themselves, but when they are compared to the moments computed by the precise equations, which provide for the effect of the axial load, the approximate methods show up as unsafe in some places and conservative in others.

One of the sources of error involved in the use of all of the approximate formulas is the fact that the ordinary three-moment equation which is used in the computation of the moments at the supports does not provide for the effect of the axial load. It would therefore appear that this error could be eliminated if the precise three-moment equation were used. So far as the moments at the points of support are concerned this is true, but a little study of the methods outlined above will show that the moments in the spans would still be in error even if the computations were modified and made to depend on the moments obtained by the *precise three-moment equation.* The approximate formulas which depend on the distance between points of inflection would give values of the maximum moments in the spans that are less than those already computed, as the effect of the greater moment at the intermediate support would be to lessen the distance between the points of inflection. This would be undesirable in the long span of the illustrative beam, as the maxima computed by the approximate formulas are already on the unsafe side in comparison with the value determined from the precise method.

The other approximate formulas would be similarly affected; the larger moment at the intermediate support as computed by the precise method would give smaller values of the maximum moments in the spans.

RESULTS OF THE STUDY OF THE APPROXIMATE FORMULAS

As a result of this study of various types of formulas for providing for the effect of an axial load in combination with a lateral load on a spar, it has been concluded that none of the approximate formulas are satisfactory for general use in the final design of a continuous or restrained spar. Johnson's formula is easy to apply and, as it gives reliable results for pin-ended struts with a lateral load, it may be used in the final design of such members.

It has been seen that the principal source of error in applying the approximate formulas to continuous members arises from the use of the ordinary three-moment equation for computing the moments at the points of support. The moments so obtained are in error, as no provision for the effect of the axial load is made in their computation. Attention has also been called to the fact that the use of the values of the moments at the supports, computed by the precise three-moment equation, in conjunction with the approximate formulas is liable to increase the difference between the computed and the actual maximum moments in the spans. Moreover, once the precise three-moment equation has been solved to determine the moments at the supports, it is far less laborious to determine the maximum moments in the spans by the precise method than by the approximate ones.

Webb & Thorne's method, which is really a modified form of the precise equations that is approximate because of the fact that the limit of the series introduced by the secondary stresses is expressed as an algebraic coefficient instead of a trigonometric function, will give satisfactory results when the axial load is not too close to the Euler load. This method will usually give satisfactory results for use in the design of airplane wing spars, but its use will in general entail more arithmetical work than the precise methods.

The final design of continuous or restrained members subjected to combined loading should never be made on the basis of the results obtained from the ordinary three-moment equation in conjunction with any of the approximate formulas unless the axial load is small or the margin of safety is great. The approximate methods will be sufficiently accurate for use in the preliminary design of airplane members subjected to combined loading, but all such members should be checked by one of the precise methods before being approved for use in the final design.

PART II—THE DERIVATION OF THE PRECISE EQUATION

INTRODUCTION

In Part I of this report various approximate formulas for the determination of stresses due to combined bending and compression were studied and compared to each other and to the precise formulas. None of the approximate formulas were found to be generally applicable, and the use of precise formulas was recommended. This part of the report gives the derivation of the precise formulas and the method of applying them to practical design.

The chief advantages of the precise formulas are as follows:

(1) The true bending moment at any section of the beam can be obtained.

(2) The total deflection of any section of the beam can be obtained.

(3) Very few assumptions are necessary. Those that are made are the ones generally made in developing beam and column formulas.

(4) As shown in Parts III and IV of this report, the deflections obtained from the formulas check experimental results in a very satisfactory manner, and much better than the approximate formulas.

(5) Once the size of the beam has been determined, the computations necessary with the precise formulas are less tedious than with the approximate formulas, if the margin of safety is not to be unnecessarily large.

The chief disadvantages of the precise formulas are as follows:

(1) The determination of the size members required and the stresses involved are dependent on each other in such a manner that the method of trial and error must be employed. This can, however, be overcome to a great extent by judicious approximations in the preliminary design. Furthermore, it is felt that the greater accuracy of the results is well worth any increased labor of computation.

(2) Many engineers are unfamiliar with these formulas. This is not considered sufficient reason for neglecting them in the face of their advantages.

(3) Special complex functions and trigonometric functions of numbers must be used. The complex functions needed and the trigonometric functions of numbers in the required range have been tabulated and are given in the appendixes to this report. These tables are arranged to obviate the necessity of transforming numbers considered as angles measured in radians to angles measured in degrees, and vice versa. The tediousness of this operation was formerly a very exasperating feature in the use of precise formulas.

(4) Previously the precise formulas were suspected because of the presence in them of trigonometric functions. This, however, was due to a misconception of the nature of these quantities. In the precise formulas, as explained in the introduction to this report, these functions have no connection whatsoever with any angles, but are the limits of certain infinite series that can be most conveniently expressed as trigonometric functions.

BASIC ASSUMPTIONS

The basic assumptions from which the precise formulas are developed are as follows:

(1) Plane cross sections remain plane and normal to the longitudinal fibers after bending.

(2) The intensity of stress is proportional to the strain throughout the member, and the ratios of stress to strain, the moduli of elasticity, are the same in tension and compression.

(3) Every longitudinal fiber is free to extend or contract under stress as if separate from the other fibers.

(4) The member is straight and homogeneous, and the cross section of the member is uniform between points of support.

(5) The axial load is applied in such a way as to develop no bending in the member due to eccentricities. For a perfectly homogeneous material this requires that the axial load be so applied as to pass through the centroid of each cross section of the undeflected member.

NOMENCLATURE

The nomenclature used is, with the exception of one or two abbreviated forms, standard in literature on mechanics and is self-explanatory with the use of the figures. The abbreviated forms are described in the appropriate places in the derivation and should cause no trouble.

The conventions used for signs are those given on page 3 of Part I of this report.

SCOPE OF DERIVATIONS

The derivation will be given in detail for the most common condition of loading encountered in airplane work, i. e., a member having a uniformly distributed lateral load in combination with an axial load causing compression. The development of the equations for various other conditions of loading will be given in such a way as to indicate the differences, so that they will serve as an aid to the designer should he want to derive a formula for a loading condition which is not included in this report. For brevity, computations involving only simple algebra or arithmetic are omitted from all of the derivations.

CASE I.—AXIALLY LOADED STRUT WITH UNIFORMLY DISTRIBUTED TRANSVERSE LOAD

Figure 3 shows a member supported at two points and subjected to a uniformly distributed lateral load, an axial compression, and moments applied at the points of support.

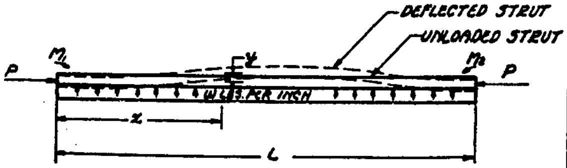

FIG. 3

The expression for the moment at any point is

$$M=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{wLx}{2}+\frac{wx^2}{2}-Py \text{-----} \quad 1$$

By making the usual assumptions of the beam theory, 1 to 3 above,

$$M=EI\frac{d^2y}{dx^2}$$

whence $EI\frac{d^2y}{dx^2}+Py=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{wLx}{2}+\frac{wx^2}{2}$

Differentiating twice with respect to x this becomes

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)+\frac{Pd^2y}{dx^2}=w$$

or

$$\frac{d^2M}{dx^2}+\frac{P}{EI}M=w$$

If we write $\frac{1}{j^2}$ for $\frac{P}{EI}$, j being an abbreviation for $\sqrt{\frac{EI}{P}}$

$$\frac{d^2M}{dx^2}+\frac{1}{j^2}M=w$$

The solution of this differential equation is[1]

$$M=C_1\sin\frac{x}{j}+C_2\cos\frac{x}{j}+wj^2 \text{----------} \quad 2$$

C_1 and C_2 are constants of integration, $\sin\frac{x}{j}$ and $\cos\frac{x}{j}$ are the limits of infinite series in which the variable is $\frac{x}{j}$. For purposes of computation they may be considered as functions of the angle $\frac{x}{j}$ expressed in radians.

The series

$$\sin\frac{x}{j}\text{ is }\frac{x}{j}-\frac{(x/j)^3}{\underline{|3}}+\frac{(x/j)^5}{\underline{|5}}-\frac{(x/j)^7}{\underline{|7}}+\ .\ .\ .$$ [2]

and

$$\cos\frac{x}{j}\text{ is }1-\frac{(x/j)^2}{\underline{|2}}+\frac{(x/j)^4}{\underline{|4}}-\frac{(x/j)^6}{\underline{|6}}+\ .\ .\ .$$

When $x=0$, $M=M_1$ and when $x=L$, $M=M_2$, hence

$$C_1=\frac{M_2-wj^2}{\sin\frac{L}{j}}-\frac{M_1-wj^2}{\tan\frac{L}{j}}=\frac{M_2-wj^2-(M_1-wj^2)\cos L/j}{\sin\frac{L}{j}}$$

$$C_2=M_1-wj^2$$

For brevity, we shall write

$$\left.\begin{aligned}D_1&=M_1-wj^2\\D_2&=M_2-wj^2\end{aligned}\right\}$$

[1] See Hudson, The Engineers' Manual, par. 363, p. 58.
[2] See Hudson, The Engineers' Manual, par. 111, p. 36.

The moment at any point is now

$$M=\left(\frac{D_2-D_1\cos L/j}{\sin L/j}\right)\sin\frac{x}{j}+D_1\cos\frac{x}{j}+wj^2 \text{----} \quad 3$$

To find the location of the section of maximum moment, differentiate equation 2, equate the first derivative to zero, and solve

$$\frac{dM}{dx}=0=\frac{C_1}{j}\cos\frac{x}{j}-\frac{C_2}{j}\sin\frac{x}{j}$$

$$\tan\frac{x}{j}=\frac{C_1}{C_2}=\frac{D_2-D_1\cos L/j}{D_1\sin\frac{L}{j}} \text{----------} \quad 4$$

The value of x determined from this equation must lie between 0 and L. Otherwise, either M_1 or M_2 is the maximum on the strut.

The maximum moment may be found by substituting the value from equation 4 in equation 3 and simplifying

$$M_{max.}=\frac{D_1}{\cos\frac{x}{j}}+wj^2 \text{--------------} \quad 5$$

The deflection at any point is found by substituting the value of M from equation 2 in equation 1 and solving, whence

$$y=\frac{1}{P}\left(M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{wLx}{2}+\frac{wx^2}{2}\right.$$

$$\left.-\frac{D_2-D_1\cos\frac{L}{j}}{\sin\frac{L}{j}}\sin\frac{x}{j}-D_1\cos\frac{x}{j}-wj^2\right) \text{----} \quad 6$$

The first derivative of equation 6 gives the slope of the tangent to the elastic curve at any point,

$$i=\frac{1}{P}\left(\frac{M_2-M_1}{L}\right)-\frac{wL}{2}+wx-\frac{C_1}{j}\cos\frac{x}{j}+\frac{C_2}{j}\sin\frac{x}{j}\Big) \text{--} \quad 7$$

If we have two continuous spans, as shown in Figure 4, the slope of the tangent at the center support will be the same for both spans, the member being continuous over this support.

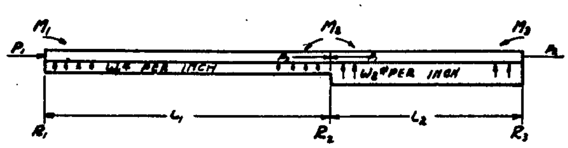

FIG. 4

At R_2, $x_1=L_1$ for the left, and $x_2=0$ for the right-hand span. Using subscripts to differentiate the symbols for the respective spans and substituting the above values in the expressions for slope at R_2, we get

$$i_1=\frac{M_2-M_1}{L_1P_1}-\frac{w_1L_1}{2P_1}+\frac{w_1L_1}{P_1}-\frac{C_1\cos\frac{L_1}{j_1}}{j_1P_1}+\frac{C_2\sin\frac{L_1}{j_1}}{j_1P_1} \text{--} \quad 8$$

where $$C_1=\frac{M_2-w_1j_1^2-(M_1-w_1j_1^2)\cos\frac{L_1}{j_1}}{\sin\frac{L_1}{j_1}}$$

$$C_2=M_1-w_1j_1^2$$

$$i_2=\frac{M_3-M_2}{L_2P_2}-\frac{w_2L_2}{2P_2}-\frac{C'_1}{j_2P_2} \text{-----------} \quad 9$$

where $$C'_1=\frac{M_3-w_2j_2^2-(M_2-w_2j_2^2)\cos\frac{L_2}{j_2}}{\sin\frac{L_2}{j_2}}$$

But at the center support $i_1 = i_2$. Substituting the values for C_1, C'_1, and C_2, equating 8 and 9, combining terms, and simplifying, the following result will be obtained:

$$\frac{M_1 L_1}{I_1}\left[\frac{\frac{L_1}{j_1}\operatorname{cosec}\frac{L_1}{j_1}-1}{\left(\frac{L_1}{j_1}\right)^2}\right]+\frac{M_3 L_2}{I_2}\left[\frac{\frac{L_2}{j_2}\operatorname{cosec}\frac{L_2}{j_2}-1}{\left(\frac{L_2}{j_2}\right)^2}\right]$$

$$+M_2\left\{\frac{L_1}{I_1}\left(\frac{\left(1-\frac{L_1}{j_1}\cot\frac{L_1}{j_1}\right)}{\left(\frac{L_1}{j_1}\right)^2}\right)\right\}+M_2\left\{\frac{L_2}{I_2}\left(\frac{\left(1-\frac{L_2}{j_2}\cot\frac{L_2}{j_2}\right)}{\left(\frac{L_2}{j_2}\right)^2}\right)\right\}$$

$$=\frac{w_1 L_1^3}{I_1}\left[\frac{\tan\frac{L_1}{2j_1}-\frac{L_1}{2j_1}}{\left(\frac{L_1}{j_1}\right)^3}\right]+\frac{w_2 L_2^3}{I_2}\left[\frac{\tan\frac{L_2}{2j_2}-\frac{L_2}{2j_2}}{\left(\frac{L_2}{j_2}\right)^3}\right] \quad \text{--- } 10$$

Multiplying this equation by 6, it becomes

$$\frac{M_1 L_1 \alpha_1}{I_1}+2M_2\left\{\frac{L_1}{I_1}\beta_1+\frac{L_2}{I_2}\beta_2\right\}+\frac{M_3 L_2 \alpha_2}{I_2}=\frac{w_1 L_1^3}{4I_1}\gamma_1 + \frac{w_2 L_2^3}{4I_2}\gamma_2 \quad \text{------ } 11$$

Where

$$\alpha=6\left[\frac{\left(\frac{L}{j}\operatorname{cosec}\frac{L}{j}-1\right)}{\left(\frac{L}{j}\right)^2}\right]$$

$$\beta=3\left[\frac{\left(1-\frac{L}{j}\cot\frac{L}{j}\right)}{\left(\frac{L}{j}\right)^2}\right]$$

$$\gamma=3\left[\frac{\left(\tan\frac{L}{2j}-\frac{L}{2j}\right)}{\left(\frac{L}{2j}\right)^3}\right]$$

Tables of sines, cosines, and tangents of L/j and of α, β and γ in terms of L/j have been computed and will be found in Appendix 3.

It often happens that moments are introduced at the points of support of continuous members due to fittings which are not concentric. The above formula may be altered to provide for this condition, as follows:

$$\frac{M_1 L_1 \alpha_1}{I_1}+2M_{-2}\left(\frac{L_1}{I_1}\beta 1\right)+2M_{+2}\left(\frac{L_2}{I_2}\beta 2\right)+\frac{M_3 L_2 \alpha_2}{I_2}=\frac{W_1 L_1^3 \gamma_1}{4 I_1}+\frac{W_2 L_2^3 \gamma_2}{4 I_2} \quad \text{------ 11a}$$

In this equation M_{-2} and M_{+2} are the moments an infinitesimal distance to the left and right of the point of support, respectively. Equation 11a contains an extra unknown which necessitates another equation for a solution. This is derived from the relation between M_{-1} and M_{+2}, M_{+2} being equal to M_{-2} plus or minus the eccentric moment M_e. Care must be taken with the sign of M_e. It should be considered positive if it increases the moment from M_{-2} to M_{+2} as one goes from left to right at the point of support.

When a truss deflects, the panel points do not necessarily lie on a straight line. Equation 11 may be modified to provide for differences in elevation of the supports, as follows:

$$\frac{M_1 L_1 \alpha_1}{I_1}+2M_2\left\{\frac{L_1}{I_1}\beta 1+\frac{L_2}{I_2}\beta 2\right\}+\frac{M_3 L_2 \alpha_3}{I_2}=\frac{6E(y_1-y_2)}{L_1}+\frac{6E(y_3-y_2)}{L_2}+\frac{W_1 L_1^3 \gamma_1}{4I_1}+\frac{W_2 L_2^3 \gamma_2}{4I_2} \quad \text{------ 11}$$

If the deflected positions of the points of support lie on straight line,

$$\frac{y_1-y_2}{L_1}=\frac{y_2-y_3}{L_2}$$

and the deflection terms drop out.

For airplane trusses y_1, y_2, and y_3 may be computed according to the usual methods for figuring truss deflection, but with the assumption that the deflection is due to the elongation of the wires alone, i. e., elongations of the spars and struts may be neglected. The deflections are usually small and may generally be omitted from the computations, though their effects should always be considered, especially for spars continuous over the center section where the deflected positions of the points of support are obviously not in a straight line.

CASE Ia.—AXIALLY LOADED STRUT WITH NO TRANSVERSE LOAD

The precise formulas above may be modified for use with struts subjected to axial loads and end moments, but no lateral load, by making $w=0$.

The position of the section of maximum moment may then be found from

$$\tan\frac{x}{j}=\frac{M_2-M_1\cos\frac{L}{j}}{M_1\sin\frac{L}{j}} \quad \text{------ 12}$$

and

$$M_{max.}=\frac{M_1}{\cos\frac{x}{j}} \quad \text{------ 13}$$

$$y=\frac{1}{P}\left[M_1+\left(\frac{M_2-M_1}{L}\right)x-\left(\frac{M_2-M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}\right)\sin\frac{x}{j}-M_1\cos\frac{x}{j}\right] \quad \text{------ 14}$$

If one end is hinged, as often happens in the case of struts used on airplane chasses, the moment at any point is

$$M=\frac{M_2\sin\frac{x}{j}}{\sin\frac{L}{j}} \quad \text{------ 15}$$

x being measured from the hinged end.

The location of the section of maximum moment is

$$x=\frac{j\pi}{2} \quad \text{------ 16}$$

$$M_{max.}=\frac{M_2}{\sin\frac{L}{j}} \quad \text{------ 17}$$

$$y=\frac{1}{P}\left[\frac{M_2 x}{L}-\frac{M_2\sin\frac{x}{j}}{\sin\frac{L}{j}}\right] \quad \text{------ 18}$$

CASE Ib.—PIN ENDED STRUT WITH AXIAL LOAD AND UNIFORMLY DISTRIBUTED TRANSVERSE LOAD

If both ends of a member subjected to an axial load and a uniformly distributed transverse load are hinged, the section of maximum moment occurs at mid span.

$$M_{max.}=wj^2\left(1-\frac{1}{\cos\frac{L}{2j}}\right)=wj^2\left(1-\sec\frac{L}{2j}\right) \quad \text{---- 19}$$

$$y=\frac{1}{P}\left[\frac{wx^2}{2}-\frac{wLx}{2}+\frac{wj^2\left(1-\cos\frac{L}{j}\right)}{\sin\frac{L}{j}}\sin\frac{x}{j}-wj^2\left(1-\cos\frac{x}{j}\right)\right] \quad \text{------------- 20}$$

Johnson's approximate formula

$$M=\frac{M_o}{1-\frac{1PL^2}{10EI}}$$

may be derived from formula 19 if the algebraic series represented by the secant be substituted for the secant term. The resulting form is

$$M=\frac{M_o}{1-\frac{5PL^2}{48EI}}$$

which reduces to Johnson's formula if 1/10 be substituted for 5/48.

CASE II.—AXIALLY LOADED STRUT WITH CONCENTRATED TRANSVERSE LOAD

Figure 5 shows a span subjected to a concentrated lateral load, an axial compression and moments applied at each support. The method of derivation of the formulas for the moment in the span and for the three-moment equation is analogous to that used in the preceding case. The determination of the constants of integration is somewhat more difficult for the concentrated than for the uniformly distributed load, but even that is simple if the procedure outlined below is followed.

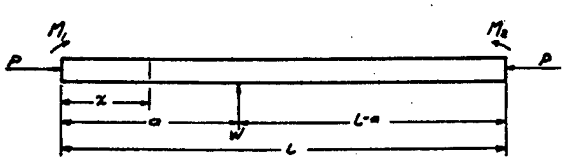

FIG. 5

Since it is impossible to write a single equation for the moment at any point in the span for this type of loading, one equation will be written for the segment to the left of the load and another for the segment to the right. The fact that two equations must be handled instead of one appears, at first glance, to render the solution much more difficult, but this will not be found to be the case.

The expression for the moment at any point between the left support and the load is

$$M_L=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{W(L-a)x}{L}-Py \quad \text{-- 21a}$$

Between the load and the right support we have

$$M_R=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{W(L-a)}{L}x+W(x-a)-Py \quad \text{------------ 21b}$$

As in Case I, take the second derivative of equation 21, whence

$$\frac{d^2M}{dx^2}=-\frac{Pd^2y}{dx^2}$$

or

$$\frac{d^2M}{dx^2}+\frac{P}{EI}M=0$$

The solution for this equation, when applied to the segment to the left of the load is

$$M_L=C_1\sin\frac{x}{j}+C_2\cos\frac{x}{j} \quad \text{--------- 22a}$$

And for the segment to the right of the load

$$M_R=C_3\sin\frac{x}{j}+C_4\cos\frac{x}{j} \quad \text{-------- 22b}$$

where $j=\sqrt{\frac{EI}{P}}$; and C_1, C_2, C_3, and C_4 are constants of integration.

From equations 21 and 22, we find that the deflection is

$$Py_L=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{W(L-a)}{L}x-C_1\sin\frac{x}{j}-C_2\cos\frac{x}{j} \quad \text{--------------- 23a}$$

and

$$Py_R=M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{W(L-a)}{L}x+W(x-a)-C_3\sin\frac{x}{j}-C_4\cos\frac{x}{j} \quad \text{--------- 23b}$$

By differentiating equations 23, we find that the slope is

$$Pi_L=\frac{M_2-M_1}{L}-\frac{W(L-a)}{L}-\frac{C_1}{j}\cos\frac{x}{j}+\frac{C_2}{j}\sin\frac{x}{j} \quad \text{-- 24a}$$

and

$$Pi_R=\frac{M_2-M_1}{L}-\frac{W(L-a)}{L}+W-\frac{C_3}{j}\cos\frac{x}{j}+\frac{C_4}{j}\sin\frac{x}{j} \quad \text{-------------- 24b}$$

From the conditions of the structure, when $x=0$, $M=M_1$; when $x=a$, $y_L=y_R$ and $i_L=i_R$; and when $x=L$, $M=M_2$. These conditions are sufficient to determine the four constants of integration, which will be found to be

$$C_1=\frac{M_2-M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}+Wj\sin\frac{a}{j}\left(\cot\frac{L}{j}-\cot\frac{a}{j}\right)$$

$$C_2=M_1$$

$$C_3=\frac{M_2-M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}+Wj\sin\frac{a}{j}\cot\frac{L}{j}$$

$$C_4=M_1-Wj\sin\frac{a}{j}$$

Differentiating equation 22 to find the section of maximum moment,

$$\frac{dM_L}{dx}=\frac{C_1}{j}\cos\frac{x}{j}-\frac{C_2}{j}\sin\frac{x}{j}=0$$

$$\tan\frac{x}{j}=\frac{C_1}{C_2}$$

Dividing equation 22 by $\cos\frac{x}{j}$, we get

$$\frac{M_L}{\cos\frac{x}{j}}=C_1\tan\frac{x}{j}+C_2$$

So

$$\frac{M_{max.}}{\cos\frac{x}{j}}=C_1\left(\frac{C_1}{C_2}\right)+C_2$$

or

$$M_{max.}=\left(\frac{C_1^2+C_2^2}{C_1}\right)\cos\frac{x}{j}=\frac{C_2}{C_1}\sqrt{C_1^2+C_2^2} \quad \text{25a}$$

Similarly, for the right-hand segment,

$$M_{max.}=\left(\frac{C_3^2+C_4^2}{C_3}\right)\cos\frac{x}{j}=\frac{C_4}{C_3}\sqrt{C_3^2+C_4^2} \quad \text{25b}$$

It will be noted that for a single concentrated load the section of maximum moment may come either to the left or right of the load, depending on the position of the load and the magnitude of the moments at the supports. It may therefore be necessary to compute the values of $\tan\frac{x}{j}$ for both segments to ascertain in which the section of maximum moment is located. If x is less than a, as found for the left-hand segment, use the formula for that segment when computing the maximum moment. If x is greater than a when computed from the values of C_1 and C_2 the section of maximum moment lies to the right of the load and its location and magnitude should be computed from C_3 and C_4. It is conceivable that the shape of the moment curves to the left and right of the load will be such that the point of zero slope of each curve will lie on the opposite side of the load from the curve itself. Since the equation for the location of the section of maximum moment is in reality simply a means of determining the point when the slope of the moment curve is zero, it will be found that for such a condition the value of x determined from C_1 and C_2 will be greater than a, while that determined from C_3 and C_4 will be less. This indicates that the maximum moment will be at the section where the load is applied and may be computed by substituting a for x; or that the concentrated load does not cause a maximum in the span, in which case either M_1 or M_2 is the maximum.

If the same procedure is followed for a continuous span having concentrated side loads as was followed in the case with the uniformly distributed side load, the following equation of three moments results:

$$\frac{M_1L_1\alpha_1}{I_1}+2M_2\left\{\frac{L_1\beta_1}{I_1}+\frac{L_2\beta_2}{I_2}\right\}+\frac{M_3L_2\alpha_2}{I_2}$$

$$=\frac{6W_1j_1^2}{I_1}\left[\frac{\sin\frac{a_1}{j_1}}{\sin\frac{L_1}{j_1}}-\frac{a_1}{L_1}\right]+\frac{6W_2j_2^2}{I_2}\left[\frac{\sin\frac{L_2-a_2}{j_2}}{\sin\frac{L_2}{j_2}}-\frac{L_2-a_2}{L_2}\right] \quad 26$$

In this equation α and β have the same values as in the case of the uniformly distributed load and may be found from the tables. It will be noted that the left-hand side of the above three-moment equation is identical to that developed for the uniform load, such differences as there are being on the right-hand side in connection with the terms which provide for the load. This equation may therefore be treated in exactly the same way as the three-moment equation in Case I to allow for the effect of an eccentric moment at one of the supports or for the deflection of the supports.

If a combined uniform and concentrated loading were applied to a continuous strut the three-moment equation could be arranged to provide for this condition by including on the right-hand side of the equation terms sufficient to provide for each load, leaving the left-hand side of the equation unaltered.

Formulas similar to those obtained in Case I may be obtained for pin-ended struts, or for struts in which one end is hinged and one restrained by substituting $M_1=M_2=0$ or $M_1=0$ in the equations.

For a pin-ended column with a concentrated lateral load in the middle of the span, the moment at any point is

$$M=C_1\sin\frac{x}{j} \quad 27$$

where

$$C_1=\frac{-Wj}{2\cos\frac{L}{2j}}$$

The maximum moment is at mid span and is

$$M_{max.}=\frac{-Wj\sin\frac{L}{2j}}{2\cos\frac{L}{2j}}=-\frac{Wj}{2}\tan\frac{L}{2j} \quad 28$$

The deflection at any point is

$$y=\frac{1}{P}\left[\frac{Wj\sin\frac{x}{j}}{2\cos\frac{L}{2j}}-\frac{Wx}{2}\right] \quad 29$$

The deflection at mid span is

$$y=\frac{Wj}{2P}\left[\tan\frac{L}{2j}-\frac{L}{2j}\right] \quad 30$$

This formula and an approximate formula derived from it are used in Part III of this report to compute the mid-span deflection of the specimens which were tested to show whether the theoretical formulas would agree with practical results. For further discussion, see Part III.

CASE III.—AXIALLY LOADED STRUT WITH UNIFORMLY VARYING LATERAL LOAD

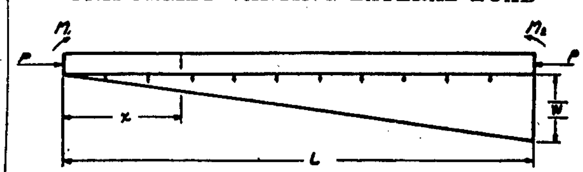

FIG. 6

Figure 6 shows a strut subjected to an axial compression, a lateral load varying uniformly from zero at one end to $\overline{W}$ at the other and to moments applied at each support. The derivation of the formulas for this condition of loading is practically identical in every step to the derivation for a uniform load. The development will therefore be given in a briefer form.

The moment at any point on the span is

$$M=M^1+\left(\frac{M_2-M_1}{L}\right)x-\frac{\overline{W}Lx}{6}+\frac{\overline{W}x^3}{6L}-Py \qquad 31$$

The second derivative of this equation is

$$\frac{d^2M}{dx^2}+P\frac{d^2y}{dx^2}=\frac{\overline{W}x}{L} \text{ or } \frac{d^2M}{dx^2}+\frac{P}{EI}M=\frac{\overline{W}x}{L}$$

The solution for this equation is

$$M=C_1\sin\frac{x}{j}+C_2\cos\frac{x}{j}+\frac{\overline{W}}{L}xj^2 \qquad 32$$

Where $j=\sqrt{\frac{EI}{P}}$, C_1 and C_2 are constants of integration.

When $x=0$, $M=M_1$ and when $x=L$, $M=M_2$, from which it will be found that

$$C_1=\frac{M_2-\overline{W}j^2-M_1\cos\frac{L}{j}}{\sin\frac{L}{j}}$$

$$C_2=M_1$$

Equation 32 does not offer a simple equation for finding the location of the section of maximum moment, as its first derivative, when equated to zero, gives

$$C_2\sin\frac{x}{j}-C_1\cos\frac{x}{j}=\frac{\overline{W}j^3}{L}$$

which may be converted into the following expression in terms of C_1, C_2, and $\overline{W}j^3$.

$$\tan\frac{x}{j}=\frac{C_1}{C_2}-\frac{(C_1^2+C_2^2)\,\overline{W}j^3}{C_1C_2\overline{W}j^3\pm C_2^2\sqrt{L^2(C_1^2+C_2^2)-(\overline{W}j^3)^2}} \qquad 33$$

It will be observed that two values of x will be found from the equation, according to the sign used for the radical term in the denominator. One value of x will probably not lie on the span, hence may be neglected.

The magnitude of the maximum moment is found by substituting the value of x obtained from equation 33 in equation 32 and solving.

The deflection at any point may be found from

$$y=\frac{1}{P}\left[M_1+\left(\frac{M_2-M_1}{L}\right)x-\frac{\overline{W}Lx}{6}+\frac{\overline{W}x^3}{6L}-C^1\sin\frac{x}{j}-C_2\cos\frac{x}{j}-\frac{\overline{W}xj^2}{L}\right] \qquad 34$$

And the slope at any point is

$$i=\frac{1}{P}\left[\frac{M_2-M_1}{L}-\frac{\overline{W}L}{6}+\frac{\overline{W}x^2}{2L}-\frac{C^1}{j}\cos\frac{x}{j}+\frac{C_2}{j}\sin\frac{x}{j}-\frac{\overline{W}j^2}{L}\right] \qquad 35$$

The three-moment equation is found in the same way as for the uniformly distributed load and may be written

$$\frac{M_1L_1\alpha_1}{I_1}+2M_2\left\{\frac{L_1}{I_1}\beta_1+\frac{L_2}{I_2}\beta_2\right\}+\frac{M_3L_2\alpha_2}{I_2}=\frac{\overline{W}_1L_1j_1^2}{I_1}[2(\beta_1-1)]+\frac{\overline{W}_2L_2j_2^2}{I_2}[\alpha_2-1] \qquad 36$$

If the load varies from $\overline{W}$ at the left support to zero at the right, the equation is similar and may be written

$$\frac{M_1L_1\alpha_1}{I_1}+2M_2\left\{\frac{L_1}{I_1}\beta_1+\frac{L_2}{I_2}\beta_2\right\}+\frac{M_3L_2\alpha_2}{I_2}=\frac{\overline{W}_1L_1j_1^2}{I_1}[\alpha_1-1]+\frac{\overline{W}_2L_2j_2^2}{I_2}[2(\beta_2-1)] \qquad 37$$

CASE IV.—GENERAL CASE

If, instead of a concentrated load W, a load of $w\,dx$ at a variable distance x from the origin had been used in Case II, the following general form of the three-moment equation would have been obtained:

$$\frac{M_1L_1\alpha_1}{E_1I_1}+\frac{2M_{-2}L_1\beta_1}{E_1I_1}+\frac{2M_{+2}L_2\beta_2}{E_2I_2}+\frac{M_3L_2\alpha_2}{E_2I_2}=\frac{6}{P_1}\int_0^{L_1}\left(\frac{\sin\frac{x_1}{j_1}}{\sin\frac{L_1}{j_1}}-\frac{x_1}{L_1}\right)w_1dx+\frac{6}{P_2}\int_0^{L_2}\left(\frac{\sin\frac{L_2-x_2}{j_2}}{\sin\frac{L_2}{j_2}}-\frac{L_2-x_2}{L_2}\right)w_2dx+\frac{6(y_1-y_2)}{L_1}+\frac{6(y_3-y_2)}{L_2} \qquad 38$$

In general, the material will be the same throughout the length of the strut, so that E_1 may be made equal to E_2, and the equation becomes

$$\frac{M_1L_1\alpha_1}{I_1}+\frac{2M_{-2}L_1\beta_1}{I_1}+\frac{2M_{+2}L_2\beta_2}{I_2}+\frac{M_3L_2\alpha_2}{I_2}=\frac{6j_1^2}{I_1}\int_0^{L_1}\left(\frac{\sin\frac{x_1}{j_1}}{\sin\frac{L_1}{j_1}}-\frac{x_1}{L_1}\right)w_1dx+\frac{6j_2^2}{I_2}\int_0^{L_2}\left(\frac{\sin\frac{L_2-x_2}{j_2}}{\sin\frac{L_2}{j_2}}-\frac{L_2-x_2}{L_2}\right)w_2dx+\frac{6E(y_1-y_2)}{L_1}+\frac{6E(y_3-y_2)}{L_2} \qquad 38a$$

The three-moment equations developed in Cases I, II, and III may be obtained from equation 38 or 38a by integrating and simplifying the resultant expressions.

CASE V.—STRUT SUBJECTED TO UNIFORMLY DISTRIBUTED LATERAL LOAD AND AN AXIAL TENSION

The foregoing formulas have all been derived for cases where the axial load caused compression in the member. This is the more important case from a structural standpoint, since an axial compression increases the bending on the strut and accelerates its failure, whereas an axial tension tends to straighten the member out and retard its failure.

For this reason it is more conservative in the latter case to compute the stresses due to the axial and transverse loads independently and to add the results.

The formulas will be given for the case of a uniformly distributed transverse load in combination with an axial tension, so that a designer may have a basis for deriving the equations for other loading conditions if he should desire to do so.

The same procedure may be used as in the case of axial compression, except that $-P$ should be substituted for P. The equation for the moment at any point on the span, instead of equation I, page 10, becomes

$$M=M_1+\left(\frac{M_2-M_1}{L}\right)\frac{x-wLx}{2}+\frac{wx^2}{2}+Py$$

and the second derivative is

$$\frac{d^2M}{dx_2}-\frac{P}{EI}M=w$$

The solution of this differential equation is

$$M=C_1 \sinh \frac{x}{j}+C_2 \cosh \frac{x}{j}-wj^2, \text{ where } j=\sqrt{\frac{EI}{P}}$$

So

$$C_1=\frac{M_2+wj^2-(M_1+wj^2)\cosh \frac{L}{j}}{\sinh \frac{L}{j}}$$

$$C^2=M^1+wj^2$$

It will be noted that the circular functions occurring in the former cases have been replaced by hyperbolic functions and that some of the signs have been changed. These changes are brought about by the change in sign of the second term in the differential equation and will require careful attention when deriving formulas for other types of loading.

The expression for the maximum moment is similar to that for axial compression.

The location of the section is at x, where

$$\tanh \frac{x}{j}=\frac{D_1 \cosh \frac{L}{j}-D_2}{D_1 \sinh \frac{L}{j}}$$

and

$$M_{\text{max.}}=\frac{D_1}{\cosh \frac{x}{j}}-wj^2$$

Where

$$D_1=M_1+wj^2$$
$$D_2=M_2+wj^2$$

The deflection at any point is

$$y=\frac{1}{P}\left(C_1 \sinh \frac{x}{j}+C_2 \cosh \frac{x}{j}-wj^2-M_1-\left(\frac{M_2-M_1}{L}\right)x+\frac{wLx}{2}-\frac{wx^2}{2}\right)$$

The slope at any point is

$$i=\frac{1}{P}\left(\frac{C_1}{j}\cosh \frac{x}{j}+\frac{C_2}{j}\sinh \frac{x}{j}-\frac{M_2-M_1}{L}+\frac{wL}{2}-wx\right)$$

The three-moment equation is similar to that for axial compression, when α, β, and γ are replaced by α_h, β_h, and γ_h, where

$$\alpha_h=\frac{6\left(1-\frac{L}{j}\operatorname{cosech}\frac{L}{j}\right)}{\left(\frac{L}{j}\right)^2}$$

$$\beta_h=\frac{3\left(\frac{L}{j}\coth \frac{L}{j}-1\right)}{\left(\frac{L}{j}\right)^2}$$

$$\gamma_h=\frac{3\left(\frac{L}{2j}-\tanh \frac{L}{2j}\right)}{\left(\frac{L}{2j}\right)^3}$$

$$\frac{M_1 L_1 \alpha_{h1}}{I_1}+2M_2\left\{\frac{L_1}{I_1}\beta_{h1}+\frac{L_2}{I_2}\beta_{h2}\right\}+\frac{M_3 L_2 \alpha_{h2}}{I_2}=\frac{w_1 L_1^3}{4I_1}\gamma_{h1}+\frac{w_2 L_2^3}{4I_2}\gamma_{h2}$$

Values of α_h, $-\beta_h$, and γ_h are tabulated in the Appendix of this report.

Values of the hyperbolic sines, cosines, or tangents may be found in almost any fairly complete set of mathematical tables.

Terms may be added to provide for the deflection of the supports in this case just as well as in the case of axial compression.

For other conditions of loading the development of the equations will be left to the designer, with the warning that particular attention must be paid to signs, especially when obtaining a solution for the differential equation.

Should the case arise where a spar is subjected to compression in one span and tension in the next, the precise three-moment equation may be altered to provide for it by using the coefficients based on circular functions with the terms relating to the span under compression and the coefficients based on the hyperbolic functions for those relating to the span in tension. The moments and stresses within the spans are then found by use of the formulas for axial compression or tension as the case may be. This condition seldom occurs in practice, although it may be found in an airplane wing spar due to the action of the drag truss stresses, which may oppose and overcome those from the lift truss in one bay but not in the next.

CRITICAL POINTS IN THE PRECISE FORMULAS

FORMULA FOR MOMENT IN ANY POINT IN THE SPAN

Each of the precise formulas is difficult to solve for certain values of L/j. In some cases the formula will take an indeterminate form such as $\frac{\infty}{\infty}$ or $\frac{0}{0}$; in others the functions vary so rapidly that reliable results can not be obtained with ordinary straight-line interpolation.

For instance, if we investigate the general expression for the bending moment at any point in a span under combined bending and compression, i. e.,

$$M = \left(\frac{D_2 - D_1 \cos \frac{L}{j}}{\sin \frac{L}{j}} \right) \sin \frac{x}{j} + D_1 \cos \frac{x}{j} + wj^2$$

we immediately see that the term in parentheses becomes infinite when $\sin L/j$ is zero. Since $\sin L/j = 0$ when $L/j = 0, \pi, 2\pi$, etc., it is apparent that these points are critical ones when applying the formulas. It is possible, however, to compute the moment for values of L/j near the critical point, plot the results, and obtain a value of the moment at the critical point from the curve drawn through the points. Figure 7 gives the results of solving the equation for several values of L/j near π and shows that the bending moment when L/j is π is about $-42{,}100$ in. lbs. in this case.

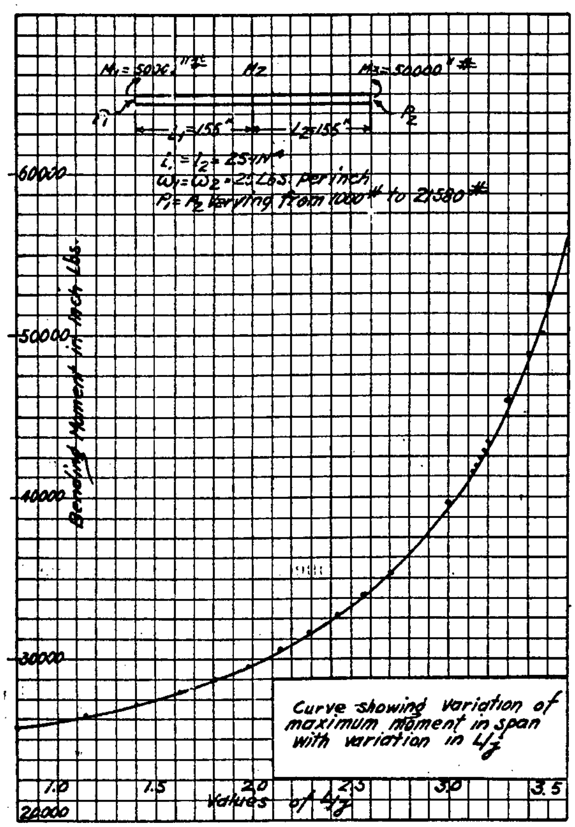

FIG. 7

FORMULA FOR MAXIMUM MOMENT IN A SPAN

If we consider the formula for obtaining the maximum moment in a span, we see that it becomes indeterminate when $D_1 = 0$. That is,

$$M_{\max.} = \frac{D_1}{\cos \frac{x}{j}} + wj^2$$

becomes $M_{\max.} = \frac{0}{0} + wj^2$ when $D_1 = 0$, since $\tan \frac{x}{j}$ as found from

$$\frac{D_2 - D_1 \cos \frac{L}{j}}{D_1 \sin \frac{L}{j}} \text{ is } \infty$$

$\frac{x}{j} = \frac{\pi}{2}$ and $\cos \frac{x}{j} = 0$. It is possible to dodge this complication by observing that $\frac{x}{j} = \frac{\pi}{2}$ and using the general expression for moment at any point in the span, which becomes

$$M = \frac{D_2}{\sin \frac{L}{j}} + wj^2$$

For points near $D_1 = 0$ the tangent is varying very rapidly and it is impossible to obtain the values of $\frac{x}{j}$ and $\cos \frac{x}{j}$ with sufficient precision by ordinary methods of interpolation.

The simplest method of attack in such a case is to interchange D_1 and D_2 in the formulas for $M_{\max.}$ In this case x will be measured from the right-hand end of the span; that is, the interchange of D_1 and D_2 is equivalent to turning the beam end for end. Unless L/j is near π in value, this dodge will alter the value of x/j sufficiently to permit a solution without recourse to any special formulas.

A second method depends on the fact that

$$\frac{1}{\cos A} = \sec A = \sqrt{\tan^2 A + 1}$$

Applying this relationship to the expression for the maximum moment in the span, we have

$$M_{\max.} = \frac{D_1}{\cos \frac{x}{j}} + wj^2 = D_1 \sqrt{\tan^2 \frac{x}{j} + 1} + wj^2$$

Substituting for $\tan \frac{x}{j}$, its value,

$$\frac{D_2 - D_1 \cos \frac{L}{j}}{D_1 \sin \frac{L}{j}}$$

and simplifying, the expression becomes

$$M_{\max.} = \frac{1}{\sin \frac{L}{j}} \sqrt{D_1^2 + D_2^2 - 2 D_1 D_2 \cos \frac{L}{j}} + wj^2$$

which does not involve the quantity x/j. By using this formula it is possible to find the maximum moment without solving for x/j, but the location of this moment is not found, and it is sometimes difficult to tell which of the two square roots of the expression under the radical sign should be used.

A third method, which is approximate but sufficiently accurate when x/j is nearly equal to $\pi/2$, depends on neglecting the 1 under the radical in the expression $\frac{1}{\cos A}=\sqrt{\tan^2 A+1}$ and assuming $\frac{1}{\cos A}=\tan A$. This is merely another way of stating the assumption that $\cos A=\cot A$, which is as valid for values of A close to $\pi/2$ as the similar assumption that $\sin A=\tan A$ for values, of A close to zero. If the above assumption is made, the formula for the maximum moment in the span becomes

$$M_{max.}=D_1 \tan\frac{x}{j}+wj^2$$

Substituting the expression for $\tan x/j$ in this formula it becomes

$$M_{max.}=\frac{D_2-D_1\cos\frac{L}{j}}{\sin\frac{L}{j}}+wj^2$$

The first two methods are simply algebraic transformations of the equations and give correct results for all values of x/j. The third method involves an error of less than 1 per cent in the first term of the expresison for $M_{max.}$ when x/j is between 1.45 and 1.70.

THREE-MOMENT EQUATION

The three-moment equation also has critical points and for certain values of L/j appears to give infinite results.

If we write the three-moment equation in a form that is readily solvable for M_2, i. e.,

$$M_2=\frac{\frac{w_1L_1^3\gamma_1}{4I_1}+\frac{w_2L_2^3\gamma_2}{4I_2}-\frac{M_1L_1\alpha_1}{I_1}-\frac{M_3L_2\alpha_2}{I_2}}{2\left\{\frac{L_1}{I_1}\beta_1+\frac{L_2}{I_2}\beta_2\right\}}$$

We find that M_2 appears to become infinite when $L/j=\pi$, since α, β, and γ are each infinite. As a matter of fact this expression really becomes $\frac{\infty}{\infty}$ and may be evaluated in the ordinary manner for solving indeterminate forms by differentiation.

It is apparent from the above equation for M_2 that the right-hand side becomes infinite when the denominator is zero. If we investigate this condition, substituting the trigonometric functions for β, we find that the denominator becomes zero when

$$\frac{1-\frac{L_1}{j_1}\cot\frac{L_1}{j_1}}{\frac{L_2}{j_2}\cot\frac{L_2}{j_2}-1}=\frac{P_1L_1}{P_2L_2}$$

Figure 8 is a curve showing the variation of moment at the intermediate point of support for different values of L/j, for the span and loading shown on the figure. It is obvious from this figure that the curve of M_2 is continuous through $L/j=\pi$.

It can also be seen from the symmetry of the loading that $P_1=P_2$, $L_1=L_2$, etc., so that in this case the moment at the center support will become infinite when

$$\frac{1-L/j\cot L/j}{L/j\cot L/j-1}=1$$

or

$$1-L/j\cot L/j=L/j\cot L/j-1$$

whence

$$L/j\cot L/j=1$$

and

$$L/j=\tan L/j$$

This condition is fulfilled when L/j is approximately

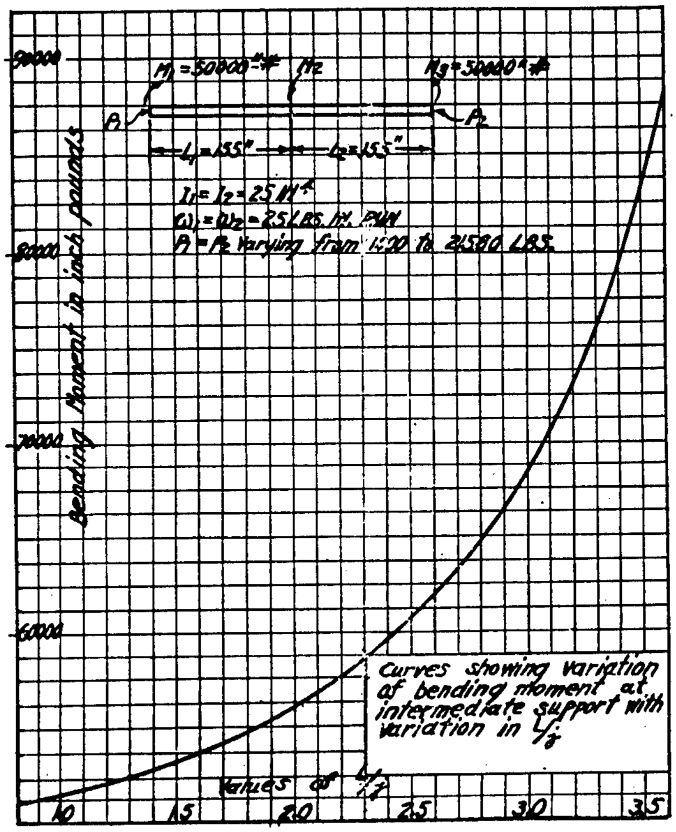

FIG. 8

4.49, as tan 4.49=4.45. The moment at the center support will not be infinite when $L/j=0$, although tan 0=0, as 0 cot 0 is not equal to 1.

When the beam is continuous over more than two spans, the problem of solving for the moments at the support when L/j in one span is equal to π becomes more involved and has not been attempted by the writers of this report.

The critical points encountered in the use of the precise formulas for a member under combined bending and compression are apparently critical only in that they require special consideration when the formulas are being solved. Similar points can be found for the other loading conditions, but they will not be enumerated here. The knowledge of their existence should be sufficient to put the designer on his guard when applying the formulas at points where the trigonometric functions are changing rapidly or at points where any of the terms in the formulas approach zero or infinity.

No difficulty will be encountered when applying these formulas so long as L/j in any span is not equal to or greater than π. When L/j is greater than π the value of the maximum moment in any span if found from the value of x/j should be checked by the other expression for the maximum moment, which is based on L/j only. There are at the present time no known test data to indicate whether or not these formulas are in reasonable agreement with actual results for loads and spans having L/j greater than π. It is therefore recommended that spar sizes to be used in airplane wings be so designed that L/j in every span will be less than π.

METHOD OF PROVIDING FOR VARIATION IN THE MOMENT OF INERTIA

In an airplane wing cellule the drag truss bays are generally shorter than those in the lift truss, there usually being two or three drag bays to one lift truss bay. The stresses in the drag or antidrag wires produce axial loads in the spars, so that the value of P can not be taken as constant throughout a lift truss bay. In addition to this variation in the axial load the spar sections themselves are often varied in the different drag bays, so that the moment of inertia is not constant. The precise formulas developed in the foregoing pages are therefore not accurate for these conditions, as they assume a constant axial load and constant moment of inertia between supports. The following method might be used for approximating the moments on the spars under such circumstances.

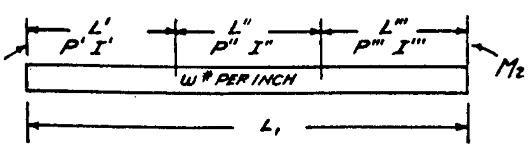

FIG. 9

Figure 9 shows one bay of a lift truss of length L_1, which is subdivided by the drag truss into three bays of length L', L'', and L'''. The axial loads are P', P'', and P''' and the cross section will be assumed to change at each drag truss panel point so that the moments of inertia are I', I'', and I'''.

Let

$$P_1=\frac{P'L'+P''L''+P'''L'''}{L'+L''+L'''}$$

and

$$I_1=\frac{I'L'+I''L''+I'''L'''}{L'+L''+L'''}$$

and compute j_1 in the usual way. This value of j should be used in the three-moment equation when computing M_1, M_2, etc. With M_1 and M_2 known, compute the moments at the drag truss panel points, i. e., M', M'', for the above value of j. Using these values of M_1, M', M'', and M_2 as the end moments, M_1 and M_2, consider each drag bay as a separate span and apply formulas 4 and 5, using the value of j obtained from the P and I for the bay being considered, not the weighted values used above. The quantities obtained in this way, while admittedly approximations, will be somewhat more conservative than those computed for the average load and average moment of inertia.

NOTES ON THE USE OF THE PRECISE FORMULAS

It is recommended that at least four significant figures be used in all computations involving the precise formulas. In preliminary investigations or for the purpose of obtaining a rough check on a spar, three figures will be sufficient but, since the final result of several of these formulas depends on small differences between large quantities, three significant figures will often give misleading results. This fact should be borne in mind and the number of significant figures necessary to give the required precision in the results should be used. This matter is especially important when the value of L/j is near π, as the functions α, β, and γ are all changing rapidly in that range.

Special care must be taken to use the proper signs throughout the computations or serious errors will result. This is particularly true in the case of the signs of the terms for loads and deflections. The conventions for signs are given on page 3 of Part I of this report.

In applying the precise formulas to design, it is necessary to decide upon a size of member before the final values of the bending moments can be obtained. This makes the process of design a matter of successive trials, but by first computing the bending moments and axial loads without allowing for secondary stresses and using those moments and loads suitably modified by the judgment of the designer on the first trial, the number of trials needed to obtain a satisfactory design should not be excessive.

In discussions of the precise methods of computing stresses due to combined loadings in this report and by other authors (as Cowley and Levy in Aeronautics in Theory and Experiment, etc.), failure is usually assumed to mean failure due to elastic instability or "buckling." Usually, before such failure would occur in practice, the member would have failed by rupture of the material due to excessive unit stresses, and statements regarding the criteria for failure must be read with these facts in mind. The criteria for failure in buckling implicitly assume that the material has a constant finite modulus of elasticity and an infinite proportional limit and ultimate allowable stress. Of course, no engineering material has such properties, but the precise formulas and resulting criteria for buckling failure are nevertheless very useful in determining the loads under which failure by rupture of the material is likely to occur.

APPENDIX I

Table A gives values of sines, cosines, and tangents of angles in terms of radians between 0.00 and 3.50 radians. This table may be used in the same way as any table of trigonometric functions, the argument being in terms of radians instead of degrees and minutes. Attention is called to the fact that the signs of the functions are given in the tables, and care should be taken to employ the correct sign when using the values in the precise equations.

Table B gives values of α, β, and γ in terms of radians between 0 and 2π. It should be noted that the increment to the argument is not constant but that it varies in different parts of the table, being least where the rate of change of α, β, and γ is greatest, so that care is required when an interpolation is made.

The functions given in Tables C are the values of α, β, and γ to be used when the axial load causes tension in the spar. They are based on the hyperbolic functions and are differentiated from the values for use with axial compressive loads by the subscript h, being written α_h, β_h, and γ_h.

Tables of hyperbolic series, cosines, and tangents are not included in this set, as they may be found in numerous collections of mathematical tables and, as the argument is practically always expressed in radians, these tables are immediately useful.

TABLE A.—*Natural sines, cosines, and tangents of angles in radians*

X in radians	X in decimals in degrees	Sine X	Cosine X	Tangent X
0.00	0.000	0.00000	1.00000	0.00000
0.01	0.573	0.01000	0.99995	0.01000
0.02	1.146	0.02000	0.99980	0.02000
0.03	1.719	0.03000	0.99955	0.03000
0.04	2.292	0.03999	0.99920	0.04002
0.05	2.865	0.04998	0.99875	0.05004
0.06	3.438	0.05996	0.99820	0.06007
0.07	4.011	0.06994	0.99755	0.07012
0.08	4.584	0.07991	0.99680	0.08017
0.09	5.157	0.08988	0.99595	0.09024
0.10	5.730	0.09983	0.99500	0.10034
0.11	6.303	0.10978	0.99396	0.11045
0.12	6.875	0.11971	0.99281	0.12057
0.13	7.448	0.12963	0.99156	0.13073
0.14	8.021	0.13954	0.99022	0.14092
0.15	8.594	0.14944	0.98877	0.15114
0.16	9.167	0.15932	0.98723	0.16138
0.17	9.740	0.16918	0.98558	0.17165
0.18	10.313	0.17903	0.98384	0.18197
0.19	10.886	0.18886	0.98200	0.19232
0.20	11.459	0.19867	0.98007	0.20271
0.21	12.032	0.20846	0.97803	0.21314
0.22	12.605	0.21823	0.97590	0.22362
0.23	13.178	0.22798	0.97367	0.23414
0.24	13.751	0.23770	0.97134	0.24472
0.25	14.324	0.24740	0.96891	0.25534
0.26	14.897	0.25708	0.96639	0.26602
0.27	15.470	0.26673	0.96377	0.27676
0.28	16.043	0.27636	0.96106	0.28756
0.29	16.616	0.28595	0.95824	0.29841

TABLE A.—*Natural sines, cosines, and tangents of angles in radians*—Continued

X in radians	X in decimals in degrees	Sine X	Cosine X	Tangent X
0.30	17.189	0.29552	0.95534	0.30934
0.31	17.762	0.30506	0.95233	0.32032
0.32	18.335	0.31457	0.94924	0.33139
0.33	18.907	0.32404	0.94604	0.34253
0.34	19.481	0.33349	0.94275	0.35374
0.35	20.054	0.34290	0.93937	0.36503
0.36	20.626	0.35227	0.93590	0.37640
0.37	21.199	0.36162	0.93233	0.38786
0.38	21.772	0.37092	0.92866	0.39941
0.39	22.345	0.38019	0.92491	0.41105
0.40	22.918	0.38942	0.92106	0.42279
0.41	23.491	0.39861	0.91712	0.43463
0.42	24.064	0.40776	0.91309	0.44657
0.43	24.637	0.41687	0.90897	0.45862
0.44	25.210	0.42594	0.90475	0.47078
0.45	25.783	0.43497	0.90045	0.48306
0.46	26.356	0.44395	0.89605	0.49545
0.47	26.929	0.45289	0.89157	0.50796
0.48	27.502	0.46178	0.88699	0.52061
0.49	28.075	0.47063	0.88233	0.53339
0.50	28.648	0.47943	0.87758	0.54630
0.51	29.221	0.48818	0.87274	0.55936
0.52	29.794	0.49688	0.86782	0.57256
0.53	30.367	0.50553	0.86281	0.58591
0.54	30.940	0.51414	0.85771	0.59943
0.55	31.513	0.52269	0.85252	0.61310
0.56	32.086	0.53119	0.84726	0.62695
0.57	32.658	0.53963	0.84190	0.64097
0.58	33.232	0.54802	0.83646	0.65517
0.59	33.805	0.55636	0.83094	0.66955
0.60	34.377	0.56464	0.82534	0.68414
0.61	34.950	0.57287	0.81965	0.69892
0.62	35.523	0.58104	0.81388	0.71391
0.63	36.096	0.58914	0.80803	0.72911
0.64	36.669	0.59720	0.80210	0.74454
0.65	37.242	0.60519	0.79608	0.76021
0.66	37.815	0.61312	0.78999	0.77611
0.67	38.388	0.62099	0.78382	0.79226
0.68	38.961	0.62879	0.77757	0.80866
0.69	39.534	0.63654	0.77125	0.82533
0.70	40.107	0.64422	0.76484	0.84229
0.71	40.680	0.65183	0.75836	0.85953
0.72	41.253	0.65938	0.75181	0.87707
0.73	41.825	0.66687	0.74517	0.89492
0.74	42.399	0.67429	0.73847	0.91309
0.75	42.972	0.68164	0.73169	0.93160
0.76	43.545	0.68892	0.72484	0.95045
0.77	44.118	0.69614	0.71791	0.96967
0.78	44.691	0.70328	0.71091	0.98926
0.79	45.264	0.71035	0.70385	1.00924
0.80	45.837	0.71736	0.69671	1.02964
0.81	46.410	0.72429	0.68950	1.05046
0.82	46.983	0.73115	0.68222	1.07171
0.83	47.556	0.73793	0.67488	1.09343
0.84	48.128	0.74464	0.66746	1.11563
0.85	48.701	0.75128	0.65998	1.13834
0.86	49.274	0.75784	0.65244	1.16155
0.87	49.847	0.76433	0.64483	1.18533
0.88	50.420	0.77074	0.63715	1.20967
0.89	50.993	0.77707	0.62941	1.23460
0.90	51.566	0.78333	0.62161	1.26016
0.91	52.139	0.78950	0.61375	1.28637
0.92	52.712	0.79560	0.60582	1.31326
0.93	53.285	0.80162	0.59783	1.34088
0.94	53.858	0.80756	0.58979	1.36923
0.95	54.431	0.81342	0.58168	1.39838
0.96	55.004	0.81919	0.57352	1.42836
0.97	55.577	0.82489	0.56530	1.45920
0.98	56.150	0.83050	0.55702	1.49096
0.99	56.723	0.83603	0.54869	1.52368

TABLE A.—*Natural sines, cosines, and tangents of angles in radians*—Continued

X in radians	X in decimals in degrees	Sine X	Cosine X	Tangent X
1. 00	57. 296	0. 84147	0. 54030	1. 55741
1. 01	57. 869	0. 84683	0. 53186	1. 59221
1. 02	58. 442	0. 85211	0. 52337	1. 62813
1. 03	59. 015	0. 85730	0. 51482	1. 66525
1. 04	59. 588	0. 86240	0. 50622	1. 70361
1. 05	60. 161	0. 86742	0. 49757	1. 74332
1. 06	60. 733	0. 87236	0. 48887	1. 78442
1. 07	61. 306	0. 87720	0. 48012	1. 82703
1. 08	61. 879	0. 88196	0. 47133	1. 87122
1. 09	62. 452	0. 88663	0. 46249	1. 91710
1. 10	63. 025	0. 89121	0. 45360	1. 96476
1. 11	63. 598	0. 89570	0. 44466	2. 01434
1. 12	64. 171	0. 90010	0. 43568	2. 06595
1. 13	64. 744	0. 90441	0. 42666	2. 11975
1. 14	65. 317	0. 90863	0. 41759	2. 17588
1. 15	65. 890	0. 91276	0. 40849	2. 23449
1. 16	66. 463	0. 91680	0. 39934	2. 29580
1. 17	67. 036	0. 92075	0. 39015	2. 35998
1. 18	67. 609	0. 92461	0. 38092	2. 42726
1. 19	68. 182	0. 92837	0. 37166	2. 49790
1. 20	68. 755	0. 93204	0. 36236	2. 57215
1. 21	69. 328	0. 93562	0. 35302	2. 65033
1. 22	69. 901	0. 93910	0. 34365	2. 73276
1. 23	70. 474	0. 94249	0. 33424	2. 81982
1. 24	71. 047	0. 94578	0. 32480	2. 91194
1. 25	71. 620	0. 94898	0. 31532	3. 00957
1. 26	72. 193	0. 95209	0. 30582	3. 11328
1. 27	72. 766	0. 95510	0. 29628	3. 22363
1. 28	73. 339	0. 95802	0. 28672	3. 34135
1. 29	73. 912	0. 96084	0. 27712	3. 46721
1. 30	74. 485	0. 96356	0. 26750	3. 60210
1. 31	75. 057	0. 96618	0. 25785	3. 74708
1. 32	75. 630	0. 96872	0. 24818	3. 90335
1. 33	76. 203	0. 97115	0. 23848	4. 07231
1. 34	76. 776	0. 97348	0. 22875	4. 25562
1. 35	77. 349	0. 97572	0. 21901	4. 45523
1. 36	77. 922	0. 97786	0. 20924	4. 67344
1. 37	78. 495	0. 97991	0. 19945	4. 91306
1. 38	79. 068	0. 98185	0. 18964	5. 17744
1. 39	79. 641	0. 98370	0. 17981	5. 47069
1. 40	80. 214	0. 98545	0. 16997	5. 79788
1. 41	80. 787	0. 98710	0. 16010	6. 16537
1. 42	81. 360	0. 98865	0. 15023	6. 58110
1. 43	81. 933	0. 99010	0. 14033	7. 05546
1. 44	82. 506	0. 99146	0. 13042	7. 60182
1. 45	83. 079	0. 99271	0. 12050	8. 23810
1. 46	83. 652	0. 99387	0. 11057	8. 98862
1. 47	84. 225	0. 99492	0. 10063	9. 88740
1. 48	84. 798	0. 99588	0. 09067	10. 98338
1. 49	85. 371	0. 99674	0. 08071	12. 34991
1. 50	85. 944	0. 99749	0. 07074	14. 10142
1. 51	86. 517	0. 99815	0. 06076	16. 42811
1. 52	87. 090	0. 99871	0. 05077	19. 6696
1. 53	87. 663	0. 99917	0. 04079	24. 4986
1. 54	88. 236	0. 99953	0. 03079	32. 4513
1. 55	88. 808	0. 99978	0. 02079	48. 0803
1. 56	89. 381	0. 99994	+0. 01080	92. 6238
1. 57	89. 954	1. 00000	+0. 00080	1, 275. 04
1. 58	90. 527	0. 99996	−0. 00920	−108. 661
1. 59	91. 100	0. 99982	−0. 01920	−52. 0676
1. 60	91. 673	0. 99957	−0. 02920	−34. 2329
1. 61	92. 246	0. 99923	−0. 03920	−25. 4950
1. 62	92. 819	0. 99879	−0. 04919	−20. 3073
1. 63	93. 392	0. 99825	−0. 05917	−16. 8712
1. 64	93. 965	0. 99760	−0. 06915	−14. 4270
1. 65	94. 538	0. 99687	−0. 07912	−12. 59926
1. 66	95. 111	0. 99602	−0. 08908	−11. 18059
1. 67	95. 684	0. 99508	−0. 09904	−10. 04724
1. 68	96. 257	0. 99404	−0. 10899	−9. 12076
1. 69	96. 830	0. 99290	−0. 11892	−8. 34925
1. 70	97. 403	0. 99167	−0. 12885	−7. 69660
1. 71	97. 976	0. 99033	−0. 13876	−7. 13723
1. 72	98. 549	0. 98889	−0. 14865	−6. 65245
1. 73	99. 122	0. 98736	−0. 15854	−6. 22809
1. 74	99. 695	0. 98572	−0. 16840	−5. 85353
1. 75	100. 268	0. 98399	−0. 17825	−5. 52037
1. 76	100. 841	0. 98215	−0. 18808	−5. 22209
1. 77	101. 414	0. 98023	−0. 19789	−4. 95340
1. 78	101. 987	0. 97819	−0. 20768	−4. 71010
1. 79	102. 559	0. 97607	−0. 21746	−4. 48866

TABLE A.—*Natural sines, cosines, and tangents of angles in radians*—Continued

X in radians	X in decimals in degrees	Sine X	Cosine X	Tangent X
1. 80	103. 132	0. 97385	−0. 22721	−4. 28627
1. 81	103. 705	0. 97152	−0. 23693	−4. 10050
1. 82	104. 278	0. 96911	−0. 24664	−3. 92937
1. 83	104. 851	0. 96659	−0. 25631	−3. 77118
1. 84	105. 424	0. 96398	−0. 26597	−3. 62450
1. 85	105. 997	0. 96127	−0. 27559	−3. 48806
1. 86	106. 570	0. 95847	−0. 28519	−3. 36083
1. 87	107. 143	0. 95557	−0. 29476	−3. 24188
1. 88	107. 716	0. 95257	−0. 30430	−3. 13039
1. 89	108. 289	0. 94949	−0. 31381	−3. 02566
1. 90	108. 862	0. 94630	−0. 32329	−2. 92710
1. 91	109. 435	0. 94302	−0. 33274	−2. 83414
1. 92	110. 008	0. 93964	−0. 34215	−2. 74630
1. 93	110. 581	0. 93618	−0. 35153	−2. 66316
1. 94	111. 154	0. 93262	−0. 36087	−2. 58433
1. 95	111. 727	0. 92896	−0. 37018	−2. 50947
1. 96	112. 300	0. 92521	−0. 37946	−2. 43828
1. 97	112. 873	0. 92137	−0. 38869	−2. 37049
1. 98	113. 446	0. 91744	−0. 39788	−2. 30582
1. 99	114. 019	0. 91339	−0. 40703	−2. 24408
2. 00	114. 592	0. 90930	−0. 41615	−2. 18504
2. 01	115. 165	0. 90509	−0. 42522	−2. 12853
2. 02	115. 738	0. 90079	−0. 43425	−2. 07437
2. 03	116. 310	0. 89641	−0. 44323	−2. 02242
2. 04	116. 883	0. 89193	−0. 45218	−1. 97252
2. 05	117. 456	0. 88736	−0. 46107	−1. 92456
2. 06	118. 029	0. 88270	−0. 46993	−1. 87841
2. 07	118. 602	0. 87797	−0. 47873	−1. 83396
2. 08	119. 175	0. 87313	−0. 48748	−1. 79112
2. 09	119. 748	0. 86822	−0. 49619	−1. 74977
2. 10	120. 321	0. 86319	−0. 50485	−1. 70984
2. 11	120. 894	0. 85812	−0. 51345	−1. 67127
2. 12	121. 467	0. 85294	−0. 52200	−1. 63395
2. 13	122. 040	0. 84768	−0. 53051	−1. 59785
2. 14	122. 613	0. 84233	−0. 53896	−1. 56287
2. 15	123. 186	0. 83690	−0. 54736	−1. 52898
2. 16	123. 759	0. 83138	−0. 55569	−1. 49610
2. 17	124. 332	0. 82579	−0. 56399	−1. 46419
2. 18	124. 905	0. 82010	−0. 57222	−1. 43321
2. 19	125. 478	0. 81434	−0. 58039	−1. 40310
2. 20	126. 051	0. 80849	−0. 58850	−1. 37382
2. 21	126. 624	0. 80258	−0. 59656	−1. 34534
2. 22	127. 197	0. 79657	−0. 60455	−1. 31761
2. 23	127. 770	0. 79048	−0. 61249	−1. 29060
2. 24	128. 343	0. 78432	−0. 62036	−1. 26429
2. 25	128. 916	0. 77807	−0. 62818	−1. 23863
2. 26	129. 489	0. 77175	−0. 63593	−1. 21360
2. 27	130. 061	0. 76536	−0. 64361	−1. 18916
2. 28	130. 634	0. 75888	−0. 65124	−1. 16531
2. 29	131. 207	0. 75232	−0. 65879	−1. 14199
2. 30	131. 780	0. 74571	−0. 66628	−1. 11921
2. 31	132. 353	0. 73902	−0. 67370	−1. 09694
2. 32	132. 926	0. 73224	−0. 68106	−1. 07514
2. 33	133. 499	0. 72538	−0. 68834	−1. 05381
2. 34	134. 072	0. 71847	−0. 69556	−1. 03292
2. 35	134. 645	0. 71147	−0. 70271	−1. 01247
2. 36	135. 218	0. 70441	−0. 70979	−0. 99242
2. 37	135. 791	0. 69728	−0. 71680	−0. 97276
2. 38	136. 364	0. 69007	−0. 72374	−0. 95349
2. 39	136. 937	0. 68281	−0. 73060	−0. 93457
2. 40	137. 510	0. 67547	−0. 73739	−0. 91602
2. 41	138. 083	0. 66806	−0. 74411	−0. 89779
2. 42	138. 656	0. 66058	−0. 75076	−0. 87989
2. 43	139. 229	0. 65304	−0. 75733	−0. 86230
2. 44	139. 802	0. 64544	−0. 76383	−0. 84502
2. 45	140. 375	0. 63777	−0. 77023	−0. 82801
2. 46	140. 948	0. 63003	−0. 77657	−0. 81130
2. 47	141. 521	0. 62224	−0. 78283	−0. 79485
2. 48	142. 094	0. 61438	−0. 78901	−0. 77866
2. 49	142. 667	0. 60646	−0. 79512	−0. 76272
2. 50	143. 240	0. 59847	−0. 80114	−0. 74703
2. 51	143. 812	0. 59043	−0. 80709	−0. 73155
2. 52	144. 385	0. 58233	−0. 81295	−0. 71632
2. 53	144. 958	0. 57417	−0. 81874	−0. 70129
2. 54	145. 531	0. 56596	−0. 82444	−0. 68647
2. 55	146. 104	0. 55769	−0. 83005	−0. 67186
2. 56	146. 677	0. 54936	−0. 83559	−0. 65744
2. 57	147. 250	0. 54097	−0. 84104	−0. 64322
2. 58	147. 823	0. 53253	−0. 84641	−0. 62917
2. 59	148. 396	0. 52405	−0. 85169	−0. 61530

Table A.—*Natural sines, cosines, and tangents of angles in radians*—Continued

X in radians	X in decimals in degrees	Sine X	Cosine X	Tangent X
2.60	148.969	0.51550	−0.85689	−0.60160
2.61	149.542	0.50691	−0.86200	−0.58806
2.62	150.115	0.49827	−0.86703	−0.57468
2.63	150.688	0.48957	−0.87198	−0.56145
2.64	151.261	0.48082	−0.87682	−0.54837
2.65	151.834	0.47204	−0.88158	−0.53544
2.66	152.407	0.46319	−0.88626	−0.52264
2.67	152.980	0.45431	−0.89084	−0.50997
2.68	153.553	0.44538	−0.89534	−0.49744
2.69	154.126	0.43640	−0.89975	−0.48502
2.70	154.699	0.42738	−0.90407	−0.47273
2.71	155.272	0.41831	−0.90830	−0.46055
2.72	155.845	0.40922	−0.91244	−0.44849
2.73	156.418	0.40007	−0.91647	−0.43653
2.74	156.990	0.39089	−0.92043	−0.42467
2.75	157.563	0.38167	−0.92430	−0.41292
2.76	158.136	0.37240	−0.92807	−0.40126
2.77	158.709	0.36310	−0.93175	−0.38970
2.78	159.282	0.35377	−0.93533	−0.37822
2.79	159.855	0.34440	−0.93882	−0.36684
2.80	160.428	0.33499	−0.94222	−0.35553
2.81	161.001	0.32555	−0.94553	−0.34431
2.82	161.574	0.31608	−0.94873	−0.33316
2.83	162.147	0.30658	−0.95184	−0.32209
2.84	162.720	0.29704	−0.95487	−0.31109
2.85	163.293	0.28748	−0.95779	−0.30014
2.86	163.866	0.27788	−0.96062	−0.28928
2.87	164.439	0.26827	−0.96335	−0.27847
2.88	165.012	0.25862	−0.96598	−0.26773
2.89	165.584	0.24895	−0.96852	−0.25704
2.90	166.158	0.23925	−0.97096	−0.24641
2.91	166.731	0.22952	−0.97330	−0.23583
2.92	167.304	0.21979	−0.97555	−0.22529
2.93	167.877	0.21002	−0.97770	−0.21481
2.94	168.450	0.20022	−0.97975	−0.20437
2.95	169.023	0.19042	−0.98170	−0.19397
2.96	169.596	0.18060	−0.98356	−0.18362
2.97	170.169	0.17076	−0.98531	−0.17330
2.98	170.741	0.16089	−0.98697	−0.16301
2.99	171.314	0.15101	−0.98853	−0.15276
3.00	171.887	0.14112	−0.98999	−0.14254
3.01	172.460	0.13121	−0.99135	−0.13235
3.02	173.033	0.12129	−0.99262	−0.12219
3.03	173.606	0.11136	−0.99378	−0.11206
3.04	174.179	0.10142	−0.99484	−0.10195
3.05	174.752	0.09146	−0.99580	−0.09185
3.06	175.325	0.08150	−0.99667	−0.08177
3.07	175.898	0.07153	−0.99744	−0.07171
3.08	176.471	0.06155	−0.99810	−0.06167
3.09	177.044	0.05156	−0.99867	−0.05164
3.10	177.617	0.04150	−0.99913	−0.04162
3.11	178.190	0.03159	−0.99950	−0.03161
3.12	178.763	0.02160	−0.99977	−0.02160
3.13	179.336	0.01160	−0.99993	−0.01160
3.14	179.909	0.00160	−1.00000	−0.00160
3.15	180.482	−0.00841	−0.99997	0.00841

Table A.—*Natural sines, cosines, and tangents of angles in radians*—Continued

X in radians	Sine X	Cosine X	Tangent X
3.16	−0.01841	−0.99983	0.01841
3.17	−0.02840	−0.99960	0.02841
3.18	−0.03840	−0.99926	0.03843
3.19	−0.04839	−0.99883	0.04845
3.20	−0.05838	−0.99830	0.05848
3.21	−0.06836	−0.99766	0.06852
3.22	−0.07833	−0.99693	0.07857
3.23	−0.08829	−0.99609	0.08864
3.24	−0.09825	−0.99516	0.09873
3.25	−0.10820	−0.99413	0.10883
3.26	−0.11814	−0.99300	0.11896
3.27	−0.12806	−0.99177	0.12912
3.28	−0.13797	−0.99044	0.13930
3.29	−0.14787	−0.98901	0.14951
3.30	−0.15774	−0.98748	0.15975
3.31	−0.16761	−0.98585	0.17002
3.32	−0.17746	−0.98412	0.18033
3.33	−0.18729	−0.98230	0.19067
3.34	−0.19711	−0.98039	0.20105
3.35	−0.20690	−0.97836	0.21148
3.36	−0.21668	−0.97624	0.22195
3.37	−0.22643	−0.97403	0.23246
3.38	−0.23616	−0.97172	0.24303
3.39	−0.24587	−0.96930	0.25365
3.40	−0.25555	−0.96680	0.26431
3.41	−0.26520	−0.96419	0.27504
3.42	−0.27482	−0.96149	0.28583
3.43	−0.28443	−0.95870	0.29668
3.44	−0.29400	−0.95581	0.30759
3.45	−0.30354	−0.95282	0.31857
3.46	−0.31306	−0.94974	0.32962
3.47	−0.32254	−0.94656	0.34074
3.48	−0.33199	−0.94328	0.35195
3.49	−0.34141	−0.93992	0.36322
3.50	−0.35077	−0.93646	0.37459

Table B

Tables of values of α, β, and γ where

$$\alpha=\frac{6(L/j \operatorname{cosec} L/j-1)}{(L/j)^2}$$

$$\beta=\frac{3(1-L/j \cot L/j)}{(L/j)^2}$$

$$\gamma=\frac{3(\tan L/2j-L/2j)}{(L/2j)^3}$$

A general relation existing between α, β, and γ is

$$\alpha+2\beta-3=\gamma(L/2j)^2$$

Table of α, β, and γ functions

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
0	1.0000		1.0000		1.0000		0
0.5	1.0300		1.0171		1.0256		0.5
1.00	1.1304		1.0737		1.1113		1.00
		0.0151		0.0085		0.0128	
1.05	1.1455		1.0822		1.1241		1.05
		0.0162		0.0090		0.0138	
1.10	1.1617		1.0912		1.1379		1.10
		0.0175		0.0097		0.0148	
1.15	1.1792		1.1009		1.1527		1.15
		0.0187		0.0105		0.0159	
1.20	1.1979		1.1114		1.1686		1.20
		0.0201		0.0111		0.0170	
1.25	1.2180		1.1225		1.1856		1.25
		0.0216		0.0120		0.0183	
1.30	1.2396		1.1345		1.2039		1.30
		0.0232		0.0128		0.0196	
1.35	1.2628		1.1473		1.2235		1.35
		0.0250		0.0137		0.0210	
1.40	1.2878		1.1610		1.2445		1.40
		0.0268		0.0147		0.0226	
1.45	1.3146		1.1757		1.2671		1.45
		0.0288		0.0158		0.0243	
1.50	1.3434		1.1915		1.2914		1.50
		0.0310		0.0169		0.0260	
1.55	1.3744		1.2084		1.3174		1.55
		0.0334		0.0182		0.0281	

Table B.—Continued

Table of α, β, and γ functions—Continued

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
1.60	1.4078	0.0361	1.2266	0.0196	1.3455	0.0303	1.60
1.65	1.4439	0.0391	1.2462	0.0211	1.3758	0.0327	1.65
1.70	1.4830	0.0422	1.2673	0.0228	1.4085	0.0353	1.70
1.75	1.5252	0.0458	1.2901	0.0246	1.4438	0.0383	1.75
1.80	1.5710	0.0498	1.3147	0.0267	1.4821	0.0416	1.80
1.85	1.6208	0.0542	1.3414	0.0290	1.5237	0.0452	1.85
1.90	1.6750	0.0593	1.3704	0.0316	1.5689	0.0493	1.90
1.95	1.7343	0.0650	1.4020	0.0345	1.6182	0.0540	1.95
2.00	1.7993	0.0713	1.4365	0.0377	1.6722	0.0593	2.00
2.05	1.8706	0.0788	1.4742	0.0415	1.7315	0.0652	2.05
2.10	1.9494	0.0872	1.5157	0.0459	1.7967	0.0722	2.10
2.15	2.0366	0.0970	1.5616	0.0508	1.8689	0.0803	2.15
2.20	2.1336	0.1085	1.6124	0.0566	1.9492	0.0895	2.20
2.25	2.2421	0.1220	1.6690	0.0635	2.0387	0.1005	2.25
2.30	2.3641	0.1380	1.7325	0.0716	2.1392	0.1137	2.30
2.35	2.5021	0.1574	1.8041	0.0813	2.2529	0.1293	2.35
2.40	2.6595	0.0872	1.8854	0.0450	2.3822	0.0716	2.40
2.425	2.7467	0.0937	1.9304	0.0482	2.4538	0.0769	2.425
2.45	2.8404	0.1009	1.9786	0.0518	2.5307	0.0827	2.45
2.475	2.9413	0.1089	2.0304	0.0560	2.6134	0.0893	2.475
2.50	3.0502	0.1180	2.0864	0.0604	2.7027	0.0966	2.50
2.525	3.1682	0.1282	2.1468	0.0656	2.7993	0.1050	2.525
2.55	3.2964	0.1397	2.2124	0.0714	2.9043	0.1143	2.55
2.575	3.4361	0.1529	2.2838	0.0779	3.0186	0.1249	2.575
2.60	3.5890	0.1680	2.3617	0.0856	3.1435	0.1372	2.60
2.625	3.7570	0.1852	2.4473	0.0942	3.2807	0.1513	2.625
2.65	3.9422	0.2054	2.5415	0.1043	3.4320	0.1676	2.65
2.675	4.1476	0.2290	2.6458	0.1161	3.5996	0.1867	2.675
2.70	4.3766	0.2568	2.7619	0.1298	3.7863	0.2093	2.70
2.725	4.6334	0.2899	2.8917	0.1469	3.9956	0.2361	2.725
2.75	4.9233	0.3297	3.0386	0.1666	4.2317	0.2685	2.75
2.775	5.2530	0.3785	3.2052	0.1912	4.5002	0.3080	2.775
2.80	5.6315	0.4387	3.3964	0.2210	4.8082	0.3568	2.80
2.825	6.0702	0.5163	3.6174	0.2600	5.1650	0.4202	2.825
2.85	6.5865	0.6094	3.8774	0.3065	5.5852	0.4948	2.85
2.875	7.1959	0.7384	4.1839	0.3711	6.0800	0.5998	2.875
2.90	7.9343	0.9096	4.5550	0.4567	6.6798	0.7386	2.90
2.925	8.8439	1.1476	5.0117	0.5758	7.4184	0.9319	2.925
2.95	9.9915	1.4930	5.5875	0.7484	8.3503	1.2113	2.95
2.975	11.4845	2.0212	6.3359	1.0127	9.5616	1.6402	2.975
3.00	+13.5057	1.0238	+7.3486	0.5127	+11.2013	0.8304	3.00
3.01	+14.5295	1.1924	+7.8613	0.5970	+12.0317	0.9671	3.01
3.02	+15.7219	1.4063	+8.4584	0.7040	+12.9988	1.1405	3.02
3.03	+17.1282	1.6834	+9.1623	0.8425	+14.1393	1.3651	3.03
3.04	+18.8117	2.0513	+10.0049	1.0265	+15.5044	1.6633	3.04
3.05	+20.8629	2.5547	+11.0314	1.2782	+17.1677	2.0711	3.05
3.06	+23.4176	3.2684	+12.3096	1.6350	+19.2388	2.6498	3.06
3.07	+26.6860	4.3301	+13.9446	2.1659	+21.8886	3.5103	3.07

TABLE B—Continued

Table of α, β, and γ functions—Continued

L/j	α	$\Delta\alpha$	β	$\Delta\beta$	γ	$\Delta\gamma$	L/j
3.08	+31.0160		+16.1105		+25.3989		3.08
		6.0084		3.0051		4.8712	
3.09	+37.0244		+19.1156		+30.2701		3.09
		8.8989		4.4503		7.2137	
3.10	+45.9234		+23.5659		+37.4839		3.10
		14.5332		7.2675		11.7808	
3.11	+60.4566		+30.8334		+49.2647		3.11
		27.9956		13.9987		22.6930	
3.12	+88.4522		+44.8321		+71.9577		3.12
		76.2965		38.1491		61.8440	
3.13	+164.7487		+82.9812		+133.8017		3.13
		1034.4142		517.2088		838.4545	
3.14	+1199.1629		+600.1900		+972.2562		3.14
		∞		∞		∞	
3.15	−227.1668		−112.9747		−183.8716		3.15
		123.4092		61.7065		100.0325	
3.16	−103.7576		−51.2692		−83.8391		3.16
		36.5229		18.2614		29.6049	
3.17	−67.2348		−33.0068		−54.2342		3.17
		17.5035		8.7527		14.1885	
3.18	−49.7313		−24.2542		−40.0458		3.18
		10.2712		5.1365		8.3263	
3.19	−39.4600		−19.1176		−31.7195		3.19
		6.7537		3.3778		5.4750	
3.20	−32.7063		−15.7398		−26.2445		3.20
		4.7788		2.3903		3.8742	
3.21	−27.9276		−13.3495		−22.3703		3.21
		3.5593		1.7807		2.8858	
3.22	−24.3683		−11.5688		−19.4845		3.22
		2.7541		1.3779		2.2330	
3.23	−21.6142		−10.1909		−17.2515		3.23
		2.1940		1.0980		1.7790	
3.24	−19.4202		−9.0929		−15.4725		3.24
		1.7890		0.8955		1.4508	
3.25	−17.6312		−8.1975		−14.0218		3.25
		1.4866		0.7443		1.2057	
3.26	−16.1447		−7.4532		−12.8161		3.26
		1.2548		0.6284		1.0178	
3.27	−14.8899		−6.8248		−11.7983		3.27
		1.0733		0.5376		0.8707	
3.28	−13.8166		−6.2872		−10.9276		3.28
		0.9285		0.4652		0.7533	
3.29	−12.8881		−5.8219		−10.1743		3.29
		0.8111		0.4066		0.6581	
3.30	−12.0770		−5.4154		−9.5162		3.30
		4.6521		2.3366		3.7784	
3.40	−7.4248		−3.0787		−5.7378		3.40
		2.0479		1.0354		1.6681	
3.50	−5.3769		−2.0433		−4.0697		3.50
		1.1477		0.5861		0.9389	
3.60	−4.2292		−1.4572		−3.1308		3.60
		0.7302		0.3785		0.6016	
3.70	−3.4990		−1.0787		−2.5292		3.70
		0.5029		0.2659		0.4179	
3.80	−2.9961		−0.8128		−2.1113		3.80
		0.3647		0.1981		0.3070	
3.90	−2.6314		−0.6147		−1.8043		3.90
		0.2744		0.1544		0.2349	
4.00	−2.3570		−0.4603		−1.5694		4.00
		0.2116		0.1248		0.1854	
4.10	−2.1454		−0.3355		−1.3840		4.10
		0.1662		0.1038		0.1498	
4.20	−1.9792		−0.2317		−1.2342		4.20
		0.1317		0.0887		0.1237	
4.30	−1.8475		−0.1430		−1.1105		4.30
		0.1046		0.0778		0.1036	
4.40	−1.7429		−0.0652		−1.0069		4.40
		0.0826		0.0696		0.0881	
4.50	−1.6603		+0.0044		−0.9188		4.50
		0.0641		0.0638		0.0757	
4.60	−1.5962		+0.0682		−0.8431		4.60
		0.0810		0.1169		0.1235	
4.80	−1.5152		+0.1851		−0.7196		4.80
		0.0238		0.1124		0.0962	
5.00	−1.4914		+0.2975		−0.6234		5.00
		0.0568		0.1520		0.0938	
5.25	−1.5842		+0.4495		−0.5296		5.25
		0.1964		0.1975		0.0733	
5.5	−1.7446		+0.6470		−0.4563		5.5
		0.4898		0.3277		0.0589	
5.75	−2.2344		+0.9747		−0.3974		5.75
		1.5111		0.8268		0.0482	
6.0	−3.7455		+1.8015		−0.3492		6.0
		25.3412		12.7331		0.0404	
6.25	−29.0867		+14.5346		−0.3088		6.25
		∞		∞		0.0048	
2π	∓∞		±∞		−0.3040		2π
		∞		∞		0.0295	
6.5	+4.1490		−2.0242		−0.2745		6.5

TABLE C

Values of α_h, β_h, and γ_h for use in the precise three-moment equation for beams subjected to a uniformly distributed load and on axial tension:

$$\alpha_h = \frac{6\left(1-\frac{L}{j}\operatorname{cosech}\frac{L}{j}\right)}{\left(\frac{L}{j}\right)^2}$$

$$\beta_h = \frac{3\left(\frac{L}{j}\coth\frac{L}{j}-1\right)}{\left(\frac{L}{j}\right)^2}$$

$$\gamma_h = \frac{3\left(\frac{L}{2j}-\tanh\frac{L}{2j}\right)}{\left(\frac{L}{2j}\right)^3}$$

It will be noted that neither $\pi/2$ nor π is a critical point for values of α_h, β_h, or γ_h.

These values are derived directly from the tables compiled by Mr. Arthur Berry in his paper, "The Calculation of Stresses in Airplane Wing Spars," published in the Transactions of the Royal Aeronautical Society, 1919, the argument X in these tables being twice the argument θ used by Berry.

L/j	α_h	$\Delta\alpha_h$	β_h	$\Delta\beta_h$	γ_h	$\Delta\gamma_h$	L/j
0.00	1.0000		1.0000		1.0000		0.00
		0.0284		0.0163		0.0244	
0.50	0.9716		0.9837		0.9756		0.50
		0.0771		0.0446		0.0664	
1.00	0.8945		0.9391		0.9092		1.00
		0.0097		0.0057		0.0083	
1.05	0.8848		0.9334		0.9009		1.05
		0.0100		0.0058		0.0087	
1.10	0.8748		0.9276		0.8922		1.10
		0.0100		0.0060		0.0089	
1.15	0.8647		0.9216		0.8833		1.15
		0.0105		0.0061		0.0090	
1.20	0.8542		0.9155		0.8743		1.20
		0.0106		0.0062		0.0092	
1.25	0.8436		0.9093		0.8651		1.25
		0.0108		0.0065		0.0094	
1.30	0.8328		0.9028		0.8557		1.30
		0.0110		0.0065		0.0096	
1.35	0.8218		0.8963		0.8461		1.35
		0.0111		0.0066		0.0097	
1.40	0.8107		0.8897		0.8364		1.40
		0.0113		0.0067		0.0098	
1.45	0.7994		0.8830		0.8266		1.45
		0.0113		0.0068		0.0099	

TABLE C—Continued

L/j	α_h	$\Delta\alpha_h$	β_h	$\Delta\beta_h$	γ_h	$\Delta\gamma_h$	L/j
1.50	0.7881		0.8762		0.8167		1.50
		0.0114		0.0068		0.0100	
1.55	0.7767		0.8694		0.8067		1.55
		0.0115		0.0069		0.0100	
1.60	0.7652		0.8625		0.7967		1.60
		0.0115		0.0070		0.0100	
1.65	0.7537		0.8555		0.7867		1.65
		0.0116		0.0070		0.0101	
1.70	0.7421		0.8485		0.7766		1.70
		0.0116		0.0070		0.0102	
1.75	0.7305		0.8415		0.7664		1.75
		0.0116		0.0071		0.0104	
1.80	0.7189		0.8344		0.7560		1.80
		0.0116		0.0071		0.0103	
1.85	0.7073		0.8273		0.7457		1.85
		0.0115		0.0071		0.0102	
1.90	0.6958		0.8202		0.7355		1.90
		0.0115		0.0071		0.0102	
1.95	0.6843		0.8131		0.7253		1.95
		0.0115		0.0071		0.0101	
2.00	0.6728		0.8060		0.7152		2.00
		0.0114		0.0071		0.0101	
2.05	0.6614		0.7989		0.7051		2.05
		0.0113		0.0071		0.0101	
2.10	0.6501		0.7918		0.6950		2.10
		0.0112		0.0071		0.0100	
2.15	0.6389		0.7847		0.6850		2.15
		0.0111		0.0070		0.0100	
2.20	0.6278		0.7777		0.6750		2.20
		0.0111		0.0070		0.0098	
2.25	0.6167		0.7707		0.6652		2.25
		0.0109		0.0070		0.0097	
2.30	0.6058		0.7637		0.6553		2.30
		0.0108		0.0069		0.0098	
2.35	0.5950		0.7568		0.6457		2.35
		0.0107		0.0069		0.0097	
2.40	0.5843		0.7499		0.6360		2.40
		0.0106		0.0069		0.0095	
2.45	0.5737		0.7430		0.6265		2.45
		0.0104		0.0068		0.0095	
2.50	0.5633		0.7362		0.6170		2.50
		0.0103		0.0067		0.0093	
2.55	0.5530		0.7295		0.6077		2.55
		0.0101		0.0067		0.0092	
2.60	0.5429		0.7228		0.5985		2.60
		0.0100		0.0066		0.0092	
2.65	0.5329		0.7162		0.5893		2.65
		0.0099		0.0065		0.0090	
2.70	0.5230		0.7097		0.5803		2.70
		0.0097		0.0065		0.0088	
2.75	0.5133		0.7032		0.5715		2.75
		0.0096		0.0065		0.0088	
2.80	0.5037		0.6967		0.5627		2.80
		0.0094		0.0064		0.0085	
2.85	0.4943		0.6903		0.5542		2.85
		0.0092		0.0063		0.0085	
2.90	0.4851		0.6840		0.5457		2.90
		0.0091		0.0062		0.0085	
2.95	0.4760		0.6778		0.5372		2.95
		0.0090		0.0062		0.0084	
3.00	0.4670		0.6716		0.5288		3.00
		0.0087		0.0061		0.0083	
3.05	0.4583		0.6655		0.5205		3.05
		0.0087		0.0060		0.0080	
3.10	0.4496		0.6595		0.5125		3.10
		0.0085		0.0059		0.0080	
3.15	0.4411		0.6536		0.5045		3.15
		0.0083		0.0060		0.0077	
3.20	0.4328		0.6476		0.4968		3.20